Modeling the relation between the intensity just-noticeable difference and loudness for pure tones and wideband noise

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A classical problem in auditory theory is the relation between the loudness $L(I)$ and the intensity just-noticeable difference (JND) $\Delta I(I)$. The intensity JND is frequently expressed in terms of the Weber fraction defined by $J(I) \equiv \Delta I / I$ because it is anticipated that this ratio should be a constant (i.e., Weber's law). Unfortunately, $J(I)$ is not a constant for the most elementary case of the pure tone JND. Furthermore it remains unexplained why Weber's law holds for wide-band stimuli. We explore this problem and related issues. The loudness and the intensity JND are defined in terms of the first and second moments of a proposed random *decision variable* called the *single-trial loudness* $\tilde{L}(I)$, namely the loudness is $L(I) \equiv \mathscr{E}\tilde{L}(I)$, while the variance of the single trial loudness is $\sigma_L^2 \equiv \mathscr{E}(\tilde{L}-L)^2$. The JND is given by the signal detection assumption $\Delta L = d'\sigma_L$, where we define the loudness JND $\Delta L(I)$ as the change in loudness corresponding to $\Delta I(I)$. Inspired by Hellman and Hellman's recent theory [J. Acoust. Soc. Am. **87**, 1255–1271 (1990)], we compare the Riesz [Phys. Rev. **31**, 867–875 (1928)] $\Delta I(I)$ data to the Fletcher and Munson [J. Acoust. Soc. Am. **5**, 82–108 (1933)] loudness growth data. We then make the same comparison for Miller's [J. Acoust. Soc. Am. **19**, 609–619 (1947)] wideband noise JND and loudness match data. Based on this comparison, we show empirically that $\Delta L(L) \propto L^{1/p}$, where $p=2$ below ≈ 5 sones and is 1 above. Since $\Delta L(I)$ is proportional to σ_L, when $p=2$ the statistics of the single-trial loudness $\tilde{L}$ are Poisson-like, namely $\sigma_L^2 \propto L$. This is consistent with the idea that the pure tone loudness code is based a neural discharge rate (not the auditory nerve). Furthermore, when $p=1$ (above about 5 sones), the internal loudness signal-to-noise ratio is constant. It is concluded that Ekman's law ($\Delta L/L$ is constant) is true, rather than Weber's law, in this loudness range. One of the main contributions of this paper is its attempt to integrate Fletcher's neural excitation pattern model of loudness and signal detection theory.
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INTRODUCTION

When modeling human psychophysics we must carefully distinguish the external *physical* variables, which we call Φ variables, from the internal *psychophysical* variables, which we refer to as Ψ variables.[1] Psychophysical modeling seeks a transformation from the Φ domain to the Ψ domain. The Φ intensity of a sound is easily quantified by direct measurement. The Ψ intensity is the loudness. The idea that loudness could be quantified was first suggested by Fechner (1966) in 1860, which raised the question of the quantitative transformation between the physical and psychophysical intensity. For a recent review of this problem, and a brief summary of its long history, see Schlauch *et al.* (1995).

An increment in the intensity of a sound that results in a *just-noticeable difference* (JND) is called an intensity JND. Fechner suggested quantifying the intensity-loudness growth transformation by counting the number of the *loudness JNDs* between two intensity values. However, after many years of work, the details of the relationship between loudness and the intensity JNDs have remained unclear (Zwislocki and Jordan, 1986; Viemeister, 1988; Plack and Carlyon, 1995).

The contribution of this paper is that it takes a fresh view of the whole problem of the intensity JND and loudness by merging the 1953 Fletcher neural excitation pattern model of loudness (Allen, 1995, 1996a) with auditory signal detection theory (Green and Swets, 1966). It is generally accepted that the intensity JND is the physical correlate of the psychological-domain uncertainty corresponding to the psychological intensity representation of a signal. Along these lines, for long duration pure tones and wideband noise, we assume that the Ψ-domain intensity is the loudness, and that the loudness JND results from loudness ''noise'' due to its stochastic representation.

To model the intensity JND we must define a *decision variable* associated with loudness and its random fluctuations. We call this loudness random decision variable the *single-trial loudness*. Accordingly we define the loudness and the loudness JND in terms of the first and second moments of the single-trial loudness, corresponding to the mean and variance of the distribution of the intensity decision variable. Because of its fundamental importance, we define the ratio of the mean loudness to the loudness standard deviation as the *loudness signal-to-noise ratio* (SNR_L).

We will show that a transformation of the Φ-domain JND data into the Ψ domain unifies tonal-stimuli JND data, which do not obey the Weber's law (''near-miss results''), and wideband noise data, for which Weber's law holds. We show that $\mathrm{SNR}_L(L)$ is functionally the same for both the tone and noise cases. To help understand these results, we introduce the concept of a near-miss to Stevens' law, which we show cancels the near–miss to Weber's law, giving the invariance in SNR_L for the tone case. Our ultimate goal in this work is to use signal detection theory to unify masking and the JND, following the 1947 outline of this problem by Miller (1947). This work has applications in speech and audio coding.

For the case of tones, we have chosen to illustrate our theoretical work using the classical intensity modulation measurements of Riesz (1928). Riesz measured the intensity JND using small, low-frequency (3-Hz), sinusoidal modulation of tones. ''Modern'' methods generally use ''pulsed'' tones which are turned on and off somewhat abruptly, to make them suitable for a two-alternative, forced-choice (2AFC) paradigm. Riesz's modulation method has a distinct advantage for characterizing the internal signal detection process, because it maintains a nearly steady-state small-signal condition within the auditory system. The interpretation of intensity JNDs is therefore simplified since the underlying stochastic processes are stationary.

An outline of the paper is as follows: After some basic definitions in Sec. I, and a review of some previous models in Sec. II, in Sec. III we explore issues surrounding the relation between the intensity JND and loudness, for the special case of tones in quiet and for wideband noise. First, we look at formulas for counting the number of intensity and loudness JNDs and we use these formulas, together with decision-theoretic principles, to relate loudness to the intensity JND. We then review the loudness-JND theory developed by Hellman and Hellman (1990), which provided the inspiration for the present work. Next, we empirically estimate the loudness SNR as a function of both intensity and loudness, using the tonal JND data of Riesz (1928) and the loudness growth function of Fletcher and Munson (1933). We then repeat this calculation for Miller's wideband noise JND and loudness data. Finally we propose a model of loudness that may be used to compute the JND. This model merges Fletcher's neural excitation pattern model of loudness with signal detection theory.

I. DEFINITIONS

We need a flexible yet clear notation that accounts for important time fluctuations and modulations that are present in the signals, such as beats and gated signals. Toward this end, we propose the following definitions. We include a definition of *masked threshold* because we view the intensity JND as a special case of the masked threshold (Miller, 1947). We include a definition of *beats* so that we can discuss their influence on Riesz's method for the measurement of intensity JNDs.

A. Basic definitions

1. Intensity

In the time domain, it is common to define the Φ *intensity* in terms of the time-integrated squared signal pressure $s(t)$, namely,[2]

$$I_s(t) \equiv \frac{1}{\varrho c T}\int_{t-T}^{t} s^2(t)\,dt, \tag{1}$$

where T is the integration time and ϱc is the specific acoustic impedance of air. The *intensity level* is defined as I_s/I_{ref}, and the *sound-pressure level* as s/s_{ref} where the reference intensity is I_{ref} or 10^{-10} μW/cm^2 and the reference pressure $s_{\mathrm{ref}}=20\mu$Pa. These two reference levels are equivalent at only one temperature,[3] but both seem to be in use.

2. Intensity of masker+probe

The JND is sometimes called ''self-masking,'' to reflect the view that it is determined by the internal noise of the auditory system. To model the JND it is useful to define a more general measure called the *masked threshold*, which is defined in the Φ domain in terms of a pressure scale factor α applied to the probe signal $p(t)$ that is then added to the masking pressure signal $m(t)$. The relative intensity of the probe and masker is varied by changing α. Setting $s(t)=m(t)+\alpha p(t)$, we denote the combined intensity as

$$I_{m+p}(t,\alpha) \equiv \frac{1}{\varrho c T}\int_{t-T}^{t} (m(t)+\alpha p(t))^2\,dt. \tag{2}$$

The unscaled probe signal $p(t)$ is chosen to have the same long-term average intensity as the masker $m(t)$, defined as I. Let $I_m(t)$ be the intensity of the masker with no probe ($\alpha=0$), and $I_p(t,\alpha)=\alpha^2 I$ be the intensity of the scaled probe signal with no masker. Thus[4]

$$I \equiv I_{m+p}(t,0)=I_m(t)=I_p(t,1).$$

3. Beats

Rapid fluctuations having frequency components outside the bandwidth of the T second rectangular integration window are very small and will be ignored. Accordingly we drop the time dependence in terms I_m and I_p. Because of beats between $m(t)$ and $p(t)$ (assuming the spectra of these signals are within a common critical band) one must proceed carefully. Slowly varying correlations between the probe and masker having frequency components within the bandwidth of the integration window may *not* be ignored, as with beats between two tones separated in frequency by a few Hz. Accordingly we keep the time dependence in the term $I_{m+p}(t,\alpha)$ and other slow-beating time-dependent terms. In the Φ domain these beats are accounted for with a probe-masker correlation function $\rho_{mt}(t)$ (Sydorenko and Allen, 1994; Green and Swets, 1966, p. 213).

4. Intensity increment $\delta I(t,\alpha)$

Expanding Eq. (2) and solving for the *intensity increment* δI we find

$$\delta I(t,\alpha) \equiv I_{m+p}(t,\alpha)-I \tag{3}$$

$$= (2\alpha\rho_{mp}(t)+\alpha^2)I, \tag{4}$$

where

$$\rho_{mp}(t)=\frac{1}{\varrho c T I}\int_{t-T}^{t} m(t)p(t)\,dt \tag{5}$$

defines a normalized cross correlation function between the masker and the probe. The correlation function must lie between -1 and 1.

5. Detection threshold

As the probe to masker ratio α is slowly increased from zero, the probe can eventually be detected. We specify the *detection threshold* as α_*, where the asterisk indicates the threshold value of α where a subject can discriminate intensity $I_{m+p}(t,\alpha_*)$ from intensity $I_{m+p}(t,0)$ 50% of the time, corrected for chance [i.e., obtain a 75% correct score in a direct comparison of the two signals (Yost, 1994; Green and Swets, 1966, p. 129)]. The quantity $\alpha_*(t,I)$ is the probe to masker rms pressure ratio at the detection threshold. It is a function of the masker intensity I and, depending on the experimental setup, time.

6. Masked threshold intensity

The *masked threshold intensity* is defined in terms of α_* as

$$I_p^*(I) \equiv I_p(\alpha_*) = \alpha_*^2 I,$$

which is the threshold intensity of the probe in the presence of the masker.

The masked threshold intensity is a function of the stimulus modulation parameters. For example, tone maskers and narrow-band noise maskers of equal intensity, and therefore approximately equal loudness, give masked thresholds that are about 20 dB different (Egan and Hake, 1950). As a second example, when using the method of beats (Riesz, 1928), the just-detectable modulation depends on the beat frequency. With ''modern'' 2AFC methods, the signals are usually gated on and off (100% modulation) (Jesteadt *et al.*, 1977). According to Stevens and Davis (p. 142, 1983)

> A gradual transition, such as the sinusoidal variation used by Riesz, is less easy to detect than an abrupt transition; but, as already suggested, an abrupt transition may involve the production of unwanted transients.

One must conclude that the *relative masked threshold* [i.e., $\alpha_*(t,I)$] is a function of the modulation conditions.

7. Ψ-domain temporal resolution

When modeling time varying psychological decision variables, the relevant integration time T is not the duration defined by the Φ-intensity Eq. (1), rather the integration time is determined in the Ψ domain. This important Ψ-domain model parameter is called *loudness temporal integration* (Yost, 1994). It was first explicitly modeled by Munson in 1947.

The Φ-domain temporal resolution (T) is critical to the definition of the JND in Riesz's experiment (see Appendix A) because it determines the measured intensity of the beats. The Ψ-domain temporal resolution plays a different role. Beats cannot be heard if they are faster than, and therefore ''filtered'' out by, the Ψ domain response. The Ψ-domain temporal resolution also impacts results for gated stimuli, such as in the 2AFC experiment, though its role is poorly understood in this case. To model the JND as measured by Riesz's method of just-detectable beats, one must know the Ψ-domain resolution duration to calculate the probe-masker effective correlation $\rho_{mp}(t)$ in the Ψ domain. It may be more practical to estimate the Ψ-domain resolution from experiments that estimate the degree of correlation, as determined by the beat modulation detection threshold as a function of the beat frequency f_b (Sydorenko and Allen, 1994).

In summary, even though Riesz's modulation detection experiment is technically a masking task, we treat it, following Riesz (1928), Miller (1947), and Littler (1965), as characterizing the intensity JND.

It follows that the Ψ-domain temporal resolution plays a key role in intensity JND and masking models.

8. The intensity JND ΔI

The intensity *just-noticeable difference* (JND) is[5]

$$\Delta I(I) \equiv \delta(t,\alpha_*), \tag{6}$$

the intensity increment at the masked threshold, for the special case where the probe signal is equal to the masking signal $[p(t)=m(t)]$. From Eq. (4) with α set to threshold α_* and $\rho_{mp}(t)=1$

$$\Delta I(I) = (2\alpha_* + \alpha_*^2)I. \tag{7}$$

An important alternative definition for the special case of the *pure-tone JND* is to let the masker be a pure tone, and let the probe be a pure tone of a slightly different frequency (e.g., a beat frequency difference of $f_b=3$ Hz). This was the definition used by Riesz in 1928. Beats are heard at $f_b=3$ Hz, and assuming the period of 3 Hz is within the passband of the Ψ-temporal resolution window, $\rho_{mp}(t)=\sin(2\pi f_b t)$ and

$$\Delta I(t,I) = [2\alpha_* \sin(2\pi f_b t) + \alpha_*^2]I. \tag{8}$$

If the beat period is less than the Ψ temporal resolution window, the beats are ''filtered'' out by the auditory brain (the effective ρ_{mn} is small) and we do not hear the beats. In this case $\Delta I(I) = \alpha_*^2 I$.

9. Internal noise

It is widely accepted that the pure tone intensity JND is determined by the *internal noise* of the auditory system (Siebert, 1965; Raab and Goldberg, 1975), and that ΔI is proportional to the standard deviation of the Ψ-domain decision variable that is being discriminated in the intensity detection task, reflected back into the Φ domain. The usual assumption, from signal detection theory, is that $\Delta I = d'\sigma_I$, where $d' \equiv \Delta I/\sigma_I$ is a constant that depends on the experimental design, and σ_I is the intensity standard deviation of the Φ-domain intensity due to Ψ-domain auditory noise (Yost, 1994).

10. Hearing threshold

The *hearing threshold* (or unmasked threshold) *intensity* may be defined as the intensity corresponding to the first (lowest intensity) JND. The hearing threshold is represented as $I_p^*(0)$ to indicate the probe intensity when the masker intensity is small (i.e., $I \to 0$). It is believed that internal noise is responsible for the hearing threshold, however, there is no reason to assume that this noise is the same as the internal noise that produces the JND.

11. Loudness L

The *loudness* L of a sound is the Ψ intensity. The *loudness growth function* $L(I)$ depends on the stimulus conditions. For example $L(I)$ for a tone and for wideband noise are not the same functions. Likewise the loudness growth function for a 100-ms tone and a 1-s tone differ. When defining a *loudness scale* it is traditional to specify the intensity, frequency, and duration of a tone such that the loudness growth function is one [i.e., $L(I_{\text{ref}}, f_{\text{ref}}, T_{\text{ref}}) = 1$ defines a loudness scale]. For the sone scale, the reference signal is a $I_{\text{ref}} = 40$ dB SPL tone at $f_{\text{ref}} = 1$ kHz with duration $T_{\text{ref}} = 1$ s. For Fletcher's LU scale the reference intensity is the hearing threshold, which means that 1 sone=975 LU (Fletcher, 1953) for a ''normal'' hearing person. In the next section we shall show that Fletcher's LU loudness scale is a more natural scale than the sone scale (the ANSI and ISO standard scales).

12. The single-trial loudness

A fundamental postulate of psychophysics is that all decision variables (i.e., Ψ variables) are random variables, drawn from some probability density function (Green and Swets, 1966, Chap. 5). For early discussions of this point see Montgomery (1935) and page 144 of Stevens and Davis (1983). To clearly indicate the distinction between random and nonrandom variables, a tilde ($\sim$) is used to indicate a random variable.[6]

We define the loudness decision variable as the *single-trial loudness* $\tilde{L}$, which is the sample loudness heard on each stimulus presentation. The loudness L is then the expected value of the single-trial loudness $\tilde{L}$

$$L(I) \equiv \mathcal{E}\tilde{L}(I). \tag{9}$$

The second moment of the single-trial loudness

$$\sigma_L^2 \equiv \mathcal{E}(\tilde{L} - L)^2 \tag{10}$$

defines the loudness *variance* σ_L^2 and *standard deviation* σ_L.

B. Derived definitions

The definitions given above cover the basic variables. However, many normalized forms of these variables are used in the literature, and these also need to be defined. These derived variables were frequently formed with the hope of finding an invariance in the data. This could be viewed as a form of modeling exercise that has largely failed (e.g., the near-miss to Weber's law). The shear number of combinations has lead to serious confusions (Yost, 1994, p. 152). Each normalized variable is usually expressed in dB, adding an additional unnecessary layer of confusion to the picture.

1. Weber fraction J

The intensity JND is frequently expressed as a *relative JND* called the *Weber fraction* defined by

$$J(I) \equiv \Delta I(I)/I. \tag{11}$$

From the signal detection theory premise that $\Delta I = d' \sigma_I$ (Yost, 1994), J is just the reciprocal of an effective signal-to-noise ratio defined as

$$\text{SNR}_I(I) \equiv I/\sigma_I(I) \tag{12}$$

since

$$J = d' \sigma_I / I = d' / \text{SNR}_I. \tag{13}$$

One conceptual problem with the Weber fraction J is that it is an *effective* noise-to-signal ratio, expressed in the Φ (physical) domain, but determined by a Ψ (psychophysical) domain mechanism (internal noise).

2. Loudness JND ΔL

Any superthreshold Ψ-domain increments may be quantified by corresponding Φ-domain increments. The *loudness JND* $\Delta L(I)$ is defined as the change in loudness $L(I)$ corresponding to the intensity JND $\Delta I(I)$. While it is not possible to measure ΔL directly, we assume that we may expand the loudness function in a Taylor series, giving

$$L(I + \Delta I) = L(I) + \Delta I \left.\frac{dL}{dI}\right|_I + \text{HOT},$$

where HOT represents *higher-order terms*, which we shall ignore. If we solve for

$$\Delta L \equiv L(I + \Delta I) - L(I) \tag{14}$$

we find

$$\Delta L = \Delta I \left.\frac{dL}{dI}\right|_I. \tag{15}$$

We call this expression the *small-JND* approximation. The above shows that the loudness JND $\Delta L(I)$ is related to the intensity JND $\Delta I(I)$ by the slope of the loudness function, evaluated at intensity I. According to the signal detection model, the standard deviation of the single trial loudness is proportional to the loudness JND, namely

$$\Delta L = d' \sigma_L. \tag{16}$$

A more explicit way of expressing this assumption is

$$\frac{\Delta L}{\Delta I} = \frac{\sigma_L}{\sigma_I}. \tag{17}$$

3. Loudness SNR

In a manner analogous to the Φ-domain SNR_I, we define the Ψ-domain loudness SNR as $\text{SNR}_L(L) \equiv L/\sigma_L(L)$. Given Eq. (16), it follows that

$$\text{SNR}_I = \nu \, \text{SNR}_L, \tag{18}$$

where ν is the slope of the log-loudness function with respect to log-intensity, namely

$$\nu(\beta)\equiv \left.\frac{dL_{\log}}{d\beta}\right|_{\beta}, \tag{19}$$

where $\beta\equiv 10\ \log_{10}(I/I_{\mathrm{ref}})$ is the *intensity level* in dB, and $L_{\log}(\beta)\equiv 10\ \log_{10}(L(10^{\beta/10}))$.

The derivation of Eq. (18) is as follows: If we express the loudness as a power law

$$L(I)=I^{\nu}$$

and let $x=\log(I)$ and $y=\log(L)$, then $y=\nu x$. If the change of ν with respect to dB SPL is small, then $dy/dx\approx\Delta y/\Delta x\approx\nu$. Since $d\ \log(y)=dy/y$ we get

$$\Delta L/L=\nu\Delta I/I. \tag{20}$$

From Eq. (17), Eq. (18) follows.

Equation (18) is important because (a) it tells us how to relate the SNRs between the Φ and Ψ domains, (b) every term is dimensionless, (c) the equation is simple, since ν is approximately constant above 40 dB SL (i.e., Stevens' law), and because (d) we are used to seeing and thinking of loudness, intensity, and the SNR, on log scales, and also the slope on log-log scales.

4. Counting JND's

While the concept of counting JNDs has been frequently discussed in the literature, starting with Fechner, unfortunately the actual counting formula (i.e., the equation) is rarely provided. As a result of a literature search, we found the formula in Nutting (1907), Fletcher (1923a), Wegel and Lane (1924), Riesz (1928), Fletcher (1929), and Miller (1947).

To derive the JND counting formula, Eq. (15) is rewritten as

$$\frac{dI}{\Delta I}=\frac{dL}{\Delta L}. \tag{21}$$

Integrating over an interval gives

$$\int_{I_1}^{I_2}\frac{dI}{\Delta I}=\int_{L_1}^{L_2}\frac{dL}{\Delta L}, \tag{22}$$

where $L_1=L(I_1)$ and $L_2=L(I_2)$. Each integral counts the total number of JND's between I_1 and I_2 (Riesz, 1928; Fletcher, 1929). For example

$$N_{12}\equiv\int_{I_1}^{I_2}\frac{dI}{\Delta I(I)} \tag{23}$$

defines N_{12}, the number of intensity JNDs between I_1 and I_2. Equivalently

$$N_{12}\equiv\int_{L_1}^{L_2}\frac{dL}{\Delta L} \tag{24}$$

defines the number of loudness JNDs between L_1 and L_2. The number of JNDs must be the same regardless of the domain (i.e., the abscissa variable), Φ or Ψ.

II. EMPIRICAL MODELS

This section reviews some earlier empirical models of the JND and its relation to loudness relevant to our development.

A. Weber's law

In 1846 it was suggested by Weber that $J(I)$ is independent of I. According to Eq. (7),

$$J(I)=2\alpha_*+\alpha_*^2.$$

If J is constant, then α_* must be constant, which we denote by $\alpha_*(\cancel{I})$ (we strike out I to indicate that α_* is not a function of intensity). This expectation, which is called Weber's law (Weber, 1988), has been successfully applied to many human perceptions. We refer the reader to the helpful and detailed review of these questions by Viemeister (1988), Johnson *et al.* (1993), and Moore (1982).

Somewhat frustrating is the empirical observation that $J(I)$ is not constant for the most elementary case of a pure tone (Riesz, 1928; Jesteadt *et al.*, 1977). This observation is referred to as *the near-miss to Weber's law* (McGill and Goldberg, 1968b). It remains unexplained why Weber's law holds as well as it does (Green, 1988, 1970, p. 721), or even why it holds at all. Given the complex and nonlinear nature of the transformation between the Φ and Ψ domains, coupled with the belief that the noise source is in the Ψ domain, it seems unreasonable that a law as simple as Weber's law, could hold in any general way. A transformation of the JND from the Φ domain to the Ψ domain might clarify the situation.

Weber's law does make one simple prediction that is potentially important. From Eq. (23) along with Weber's law $J_0\equiv J(\cancel{I})$ we see that the formula for the number of JNDs is

$$N_{12}=\int_{I_1}^{I_2}\frac{dI}{J_0 I} \tag{25}$$

$$=\frac{1}{J_0}\ \ln\left(\frac{I_2}{I_1}\right). \tag{26}$$

B. Fechner's postulate

In 1860 Fechner postulated that the loudness JND $\Delta L(I)$ is a constant[7] (Stevens, 1951; Fechner, 1966; Luce, 1993; Plack and Carlyon, 1995). We shall indicate such a constancy with respect to I as $\Delta L(\cancel{I})$ (as before, we strike out the I to indicate that ΔL is *not* a function of intensity). As first reported by Stevens (1961), we shall show that Fechner's postulate is not generally true.

1. The Fechner JND counting formula

From Eq. (24), along with Fechner's postulate $\Delta L(\cancel{I})$, we find

$$N_{12}=\int_{L_1}^{L_2}\frac{dL}{\Delta L(\cancel{I})} \tag{27}$$

$$=\frac{L_2-L_1}{\Delta L}. \tag{28}$$

This says that if the loudness JND were constant, one could calculate the number of JNDs by dividing the length of the interval by the step size. We call this relation the *Fechner JND counting formula*.

2. The Weber–Fechner law

It is frequently stated (Luce, 1993) that Fechner's postulate [$\Delta L(\not I)$] and Weber's law [$J_0 \equiv J(\not I)$] lead to the conclusion that the difference in loudness between any two intensities I_1 and I_2 is proportional to the logarithm of the ratio of the two intensities, namely

$$\frac{L(I_2)-L(I_1)}{\Delta L}=\frac{1}{J_0}\log\left(\frac{I_2}{I_1}\right). \tag{29}$$

This is easily seen by eliminating N_{12} from Eq. (26) and (28). This result is called *Fechner's law*, and was called the *Weber–Fechner law* by Fletcher and his colleagues (as it is today by the Vision community) because Eq. (29) results when one assumes that both Fechner's postulate and Weber's law are simultaneously true.

Even though Weber's law is approximately true, because Fechner's postulate Eq. (28) is not true[8] (Stevens, 1961), Fechner's law cannot be true. The arguments on both sides of this proposal have been weakened by the unclear relation between loudness and the intensity JND. For example, it has been argued that since the relation between $L(I)$ and $\Delta I(I)$ depends on many factors, there can be no simple relation between the two (Zwislocki and Jordan, 1986). It has even been suggested that loudness and the intensity JND may be independent.[9] For a recent discussion of loudness and psychophysical scaling, see Marks (1974), Gescheider (1976), Luce (1993), and Plack and Carlyon (1995).

C. Poisson noise

Starting in 1923, Fletcher and Steinberg studied loudness coding of pure tones, noise, and speech (Fletcher, 1923a, 1923b; Fletcher and Steinberg, 1924; Steinberg, 1925), and proposed that loudness was related to neural spike count (Fletcher and Munson, 1933), and even provided detailed estimates of the relation between the number of spikes and the loudness in sones (Fletcher, 1953, p. 271). In 1943 De Vries first introduced a photon counting Poisson process model as a theoretical basis for the threshold of vision (De Vries, 1943). Siebert (1965) proposed that Poisson-point-process noise, resulting from the neural rate code, acts as the internal noise that limits the frequency JND (Green, 1970; Jesteadt *et al.*, 1977). A few years later (Siebert, 1968), and independently[10] McGill and Goldberg (1968a) proposed that the Poisson internal noise (PIN) model might account for the intensity JND, but they did not find a reasonable loudness growth function. Hellman and Hellman (1990) further refined the argument that Poisson noise may be used to relate the loudness growth to the intensity JND, and they found good agreement between the JND and realistic loudness functions.

As we shall show, the PIN model requires that $\Delta L(L) \propto \sqrt{L}$, which may be written as $\sigma_L^2 \propto L$. The proportionality constant depends on the loudness scale.

D. Hellman and Hellman's alternative to Fechner

In 1990 Hellman and Hellman proposed an alternative to Fechner's hypothesis, that ΔL is constant, by showing that the PIN model could give reasonable loudness growth functions. Their paper concludes that the relation between the intensity JND and loudness is

$$\sqrt{L(I_2)}-\sqrt{L(I_1)}=\frac{h}{2}\int_{I_1}^{I_2}\frac{dI}{\Delta I(I)}. \tag{30}$$

In the next section we discuss the underlying principles behind Eq. (30), and discuss its generalization to other conditions, such as higher intensities, noise, complex tones, and pulsed signals of various duty cycles.

The PIN JND counting formula. Given the definition of the number of JNDs [Eq. (23)] we may rewrite the Hellman and Hellman formula [Eq. (30)] as

$$N_{12}=\frac{2}{h}(\sqrt{L_2}-\sqrt{L_1}). \tag{31}$$

We call this relation the *PIN JND counting formula*. It specifies the number of JNDs between two loudness values, where the factor h depends on the reference intensity I_{ref} for the loudness scale. Equation (31) was first described by Stevens in 1936 (Stevens and Davis, 1983, p. 149) in a slightly modified form as $L_2=N_{02}^{2.2}$, where $L_0=0$ is the loudness for $I_0=0$, and again by Miller (1947) for white noise as $L_2=N_{02}^{3}$. Equation (31) (i.e., $L_2\approx N_{02}^2$) should be compared and contrasted to Fechner's JND counting formula Eq. (28). In the next section we show that for long duration tones, below about 20 dB SL, Eq. (31) is essentially correct; however, when the PIN model does not hold [e.g., when $\Delta L(L)\neq\sqrt{L}$, such as for continuous tones at high intensities], a different relation must apply.

III. RESULTS

In the following we directly compare the loudness-growth function of Fletcher and Munson to the number of JNDs N_{12} from Riesz, as described in Appendix A. The Fletcher and Munson loudness data (Munson, 1932) were determined for long duration tonal stimuli using the loudness balance method (Fletcher and Munson, 1933), the method of *constant stimuli* (Yost, 1994), and the assumption of additivity of partial loudness. Riesz's data were also determined for long duration stimuli with just-detectable modulation (i.e., they were tonelike sounds). Since the JND depends on the modulation depth, as discussed in the *Definitions* section, Riesz's JND data seem to be ideal for this comparison since both the loudness data and the JND data have minimal (and similar) modulation parameters (Riesz's continuous tonal stimuli, which have just-detectable modulations, are more tonelike than gated 2AFC stimuli).

A. Determination of the JND counting formula

Motivated by Eq. (31), in Fig. 1 we have compared the number of JNDs to the square root of the loudness at all 11 frequencies that Fletcher and Munson used to define the loudness, requiring the reconstruction of the loudness curves from the raw data given in Table I of the 1933 paper. The

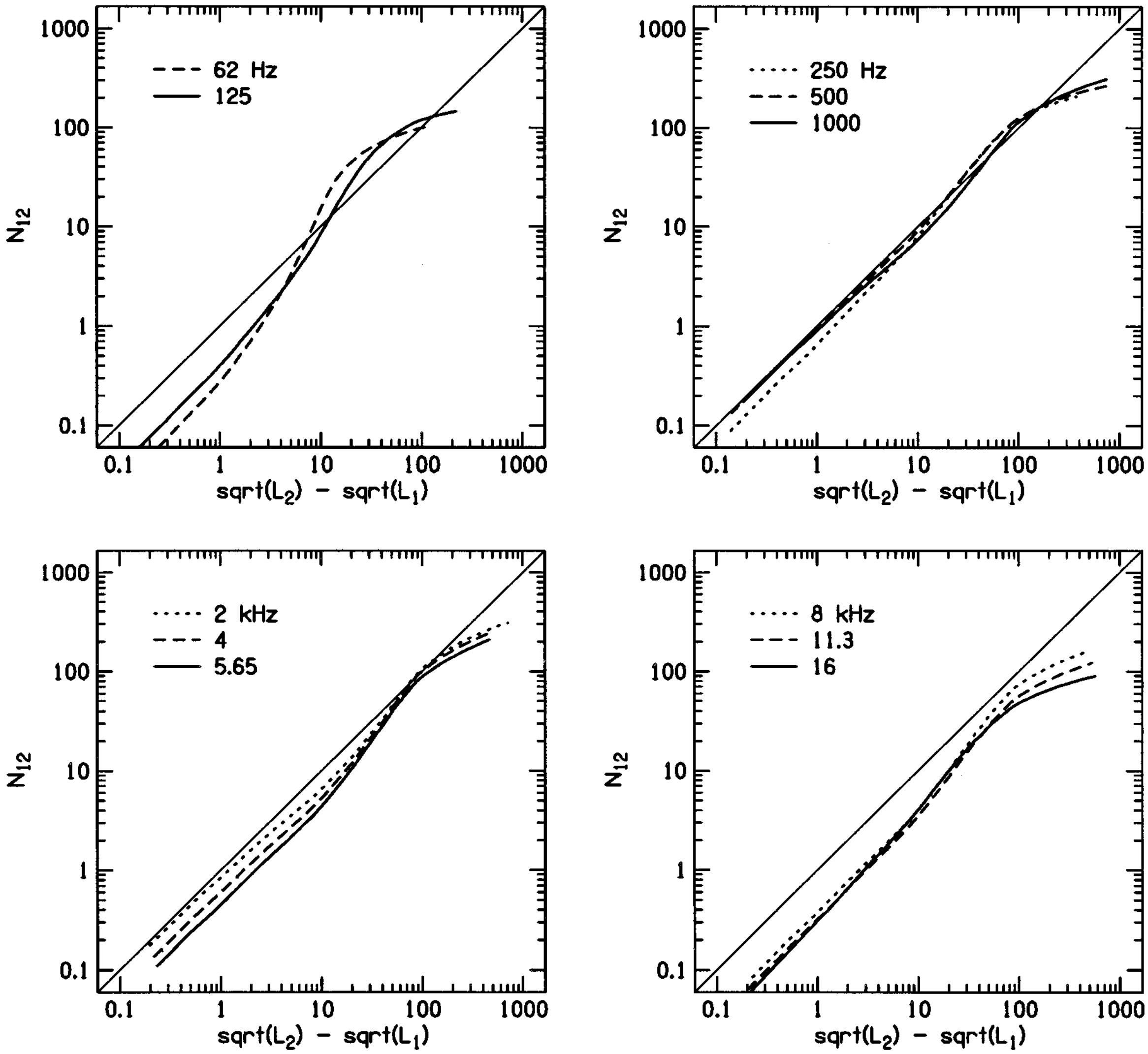

FIG. 1. Observed versus predicted number of JNDs. In this figure $\sqrt{L_2}-\sqrt{L_1}$ is the abscissa, using the loudness $L(I)$ from Fletcher and Munson (1933), versus the number of JNDs N_{12} from Riesz (1928) on the ordinate, for intensities I_2 from 1 dB to 120 dB SL (above the threshold intensity I_1). The 11 curves, corresponding to the frequencies 0.062 to 16 kHz, are distributed among the four panels for clarity. Except at low frequencies and high levels, the resulting plots are nearly parallel to the 45° line, in support of the Hellman and Hellman PIN model.

procedure for doing this is described in Appendix B. The figure is divided into four panels to separate the results across frequency. The abscissa gives the difference between the square root of loudness above threshold and the square root of the loudness at threshold $\sqrt{L_2}-\sqrt{L_1}$, while the ordinate gives the corresponding number of JNDs above threshold N_{12}. The results for 62 and 125 Hz clearly depart from straight-line behavior. Also at high levels for all frequencies, for $L>10^4$ LU (i.e., >10 sones), the results deviate from a straight-line. However, over the rest of the range, Eq. (31) is an excellent summary of the curves of Fig. 1.

Figure 2 shows an alternative way of presenting the data that estimates $2/h$ and provides a more sensitive indication of the deviations from Eq. (31). In this figure we plot the ratio of N_{12} divided by $\sqrt{L_2}-\sqrt{L_1}$, as a function of the intensity expressed in dB SL. Equation (31) says that this ratio should be independent of intensity. The deviation from a constant value shown in Fig. 2 is greatest at low frequencies, but is small in comparison to the large range of values spanned by both the numerator and denominator of this ratio. Again we see reasonable agreement between the Hellman and Hellman theory and the tonal data.

B. An alternative to Fechner's postulate

If one treats Eq. (31) as an exact representation of Fig. 2, thereby ignoring any deviations with intensity and frequency, one may draw several interesting conclusions. First it follows that

$$\Delta L = h\sqrt{L}, \tag{32}$$

where h is a proportionality constant, as may be seen by direct substitution of Eq. (32) into the JND counting formula Eq. (24):

$$N_{12} = \int_{L_1}^{L_2} \frac{dL}{h\sqrt{L}} \tag{33}$$

$$= \frac{2}{h}(\sqrt{L_2}-\sqrt{L_1}), \tag{34}$$

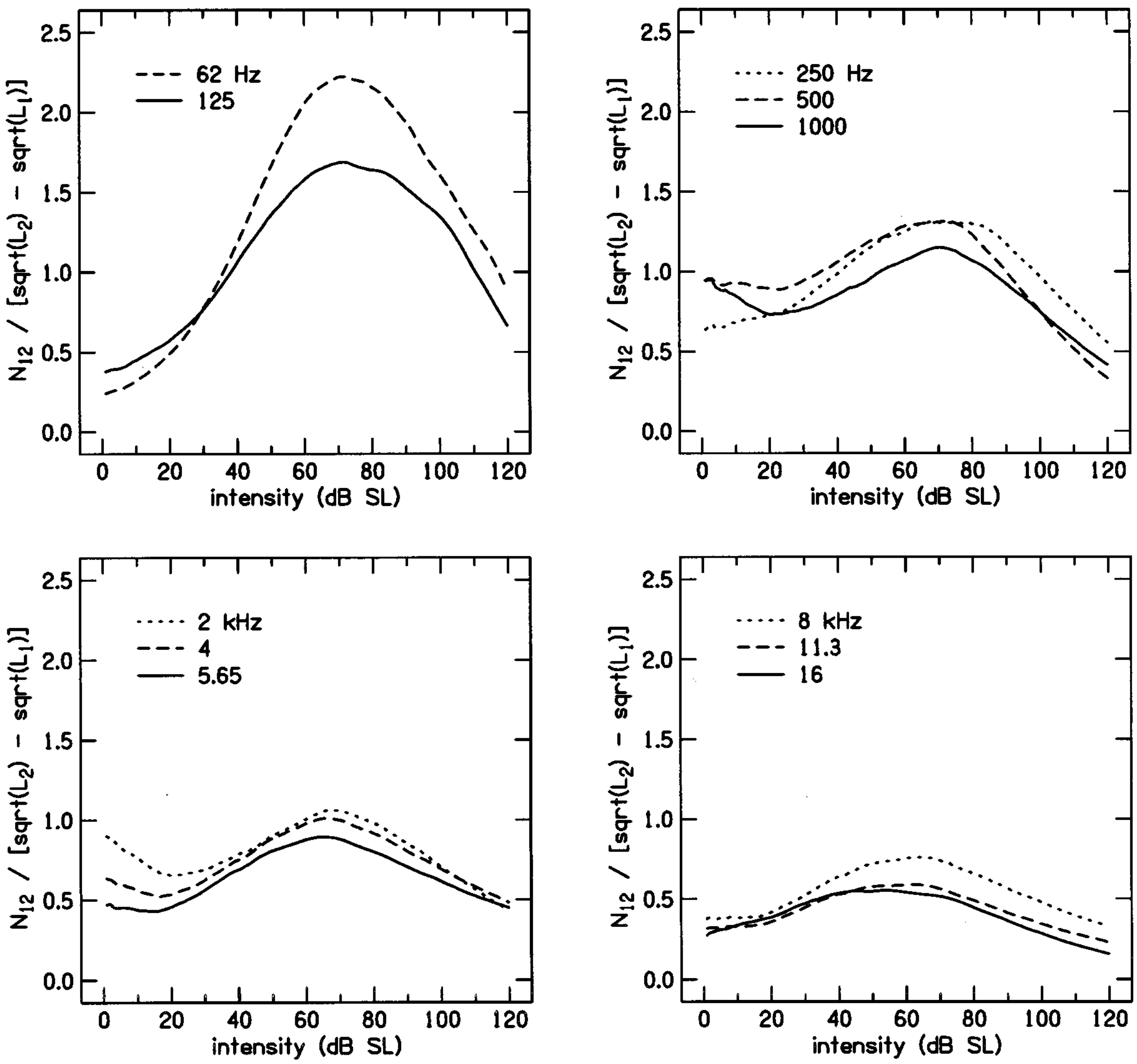

FIG. 2. These data are the same as that of Fig. 1 except the ordinate has been normalized by the abscissa. On the abscissa is the sound intensity, in dB SL, and on the ordinate is $N_{12}/(\sqrt{L_2}-\sqrt{L_1})$. This allows the estimation of the parameter $2/h$, as described by Eq. (31). The 11 curves represent the same frequencies shown in the previous figure.

which is Eq. (31). In summary, Fechner's postulate cannot be true since $\Delta L = h\sqrt{L}$, which is *not* constant.

As discussed in Sec. I, a basic tenant of signal detection theory is that the standard deviation of the decision variable is proportional to the change in the mean, which is Eq. (16) in the present case, since the decision variable is the single-trial loudness.

If we eliminate ΔL from Eqs. (32) and (16), we recover the fundamental assumption of the PIN model (Sec. II C),

$$L = \left(\frac{d'}{h}\right)^2 \sigma_L^2, \tag{35}$$

which says that *the mean of the single-trial loudness L is proportional to σ_L^2, the variance of the single-trial loudness*, where the *single-trial loudness* is the loudness decision variable. By the proper choice of the reference intensity I_{ref} corresponding to unity loudness (i.e., $L_{\mathrm{ref}}=1$), along with knowledge of d', which depends on the experimental conditions, the proportionality constant (d'/h) may be set to 1. In fact, Fletcher's LU loudness scale, which is based on spike counts, is just such a scale (Fletcher and Munson, 1933) since $L(I_{\mathrm{ref}})=1$ when I_{ref} is the threshold intensity $I_p^*(0)$. The sone scale is not such a loudness scale since in that case I_{ref} corresponds to 40 dB SPL.

Since for the PIN model we know both $\Delta I(I)$ and $\Delta L(L)$, we may evaluate Eq. (22) and obtain a usable alternative to Fechner's ill-founded loudness law Eq. (29). For example based on Eqs. (32) and (A1), Eq. (22) gives $L(I)$ for tones by equating Eqs. (31) and (A5), leading to

$$L(I) = \left[\sqrt{L_{\mathrm{ref}}} + \frac{h}{2\kappa J_\infty} \ln\left(\frac{(I/I_{\mathrm{ref}})^\kappa J_\infty + (J_{\mathrm{ref}} - J_\infty)}{J_{\mathrm{ref}}}\right)\right]^2. \tag{36}$$

The parameters κ, J_∞, and J_{ref} are described in Appendix A. Equation (36) provides a good description of the tonal loudness functions over the range of intensities where the PIN model is valid. A similar use of Eq. (22) should give a reasonable fit to any loudness growth function once $\Delta I(I)$ and $\Delta L(L)$ have been estimated.

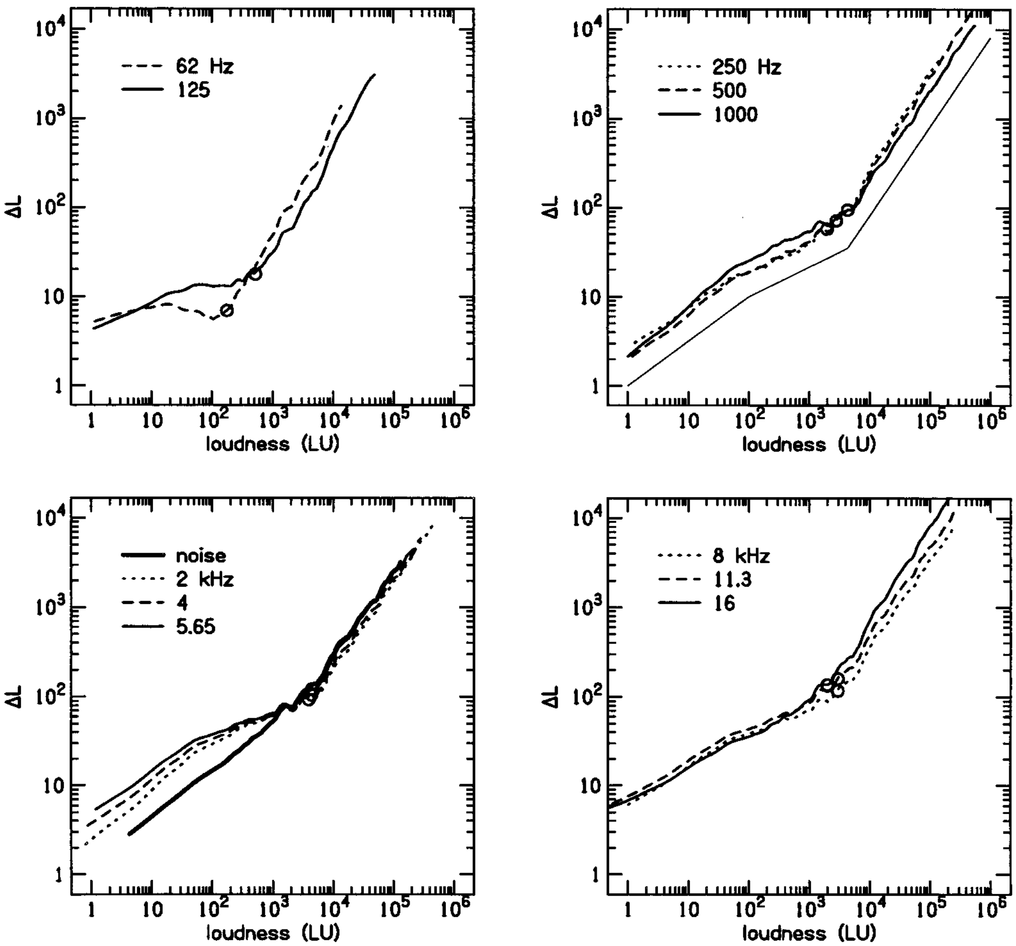

FIG. 3. In this figure we show $\Delta L(L,f)$ computed directly from Eq. (14) using Riesz's JND data and the Fletcher–Munson loudness-intensity curve, for levels between 0 and 120 dB SL. A $\odot$ has been placed on the curves at an intensity of 55 dB SL, for 62 and 125 Hz, 60 dB SL for 0.25 to 1 kHz, 55 dB SL for 2–5.65 kHz, and 50 dB SL for 8–16 kHz. In the upper-right panel we have added a straight line for reference, having slopes of 1/2, 1/3, and 1, for levels between 0–20, 20–60, and above 60 dB SL, respectively. From these plots it is clear that $\Delta L(L)$ is described by a power law in L having three straight line segments. Between 0 and 20 dB SL, the slope is close to 0.5. Between 20 and 60 dB SL the slope is close to 1/3 ($\Delta L \propto L^{1/3}$) for tones, and 2/3 for noise. Above 60 dB SL, the slope is 1 ($\Delta L \propto L$). Fechner's law $[\Delta L(\not{L})]$ appears to hold only for 62 and 125 Hz below 50 dB SL. One extra curve, labeled with a thick solid line, has been added to the lower-left panel, showing $\Delta L(L)$ for the wideband noise data of Miller (1947). This curve has a slope of approximately 1/2 below 25 dB SL, 2/3 between 25 and 55 dB SL, and then merges with the tone data up to a loudness of 10^5 (LU), the upper limit of Miller's data. Note that 1 sone is 975 LU.

C. The direct estimate of ΔL

The above discussion has (a) drawn out the fundamental nature of the JND, (b) shown the critical nature of the dependence of $\Delta L(L)$ on L, and (c) has shown that below 10 sones the PIN model, Eq. (32), approximately holds. Given its importance, it is reasonable to estimate ΔL directly from its definition Eq. (14), using Riesz's $\Delta I(I)$ and Fletcher and Munson's (1933) estimate of $L(I)$.

In Fig. 3 we show an estimate of $\Delta L(L)$ computed using all 11 tonal frequencies that Fletcher and Munson used to define the loudness. Each of the four panels displays a different frequency range. As indicated in the figure caption we have marked the point on the curve where the slope changes. For the 62 Hz data in the upper-left panel we see that ΔL is constant for levels below about 50 dB SL. Over most of the frequency range, below 20 dB SL, $\Delta L \propto \sqrt{L}$. Between 20 and 60 dB SL, $\Delta L \propto L^{1/3}$. Above 60 dB SL, $\Delta L \propto L$.

Miller's (1947) famous JND paper also includes wideband noise loudness-level results. We transformed these data to loudness using Fletcher and Munson's (1933) reference curve (i.e., Fig. 6 upper left). In Fig. 3 (thick line, lower-left panel) we show $\Delta L(L)$ for Miller's (1947) wideband noise JND_I data. Between 25 and 55 dB SL, the slope of $\Delta L(L)$ on a log–log plot is close to 2/3. Above 55 dB SL, $\Delta L(L)$ is the same as that for tones.

D. Determination of the loudness SNR

The pure tone and wideband noise JND results may be summarized in terms of the loudness $\mathrm{SNR}_L(I)$ data shown in Fig. 4 where we show $L/\Delta L = \mathrm{SNR}_L/d'$ as a function of intensity. As before we separate frequencies into separate panels. The SNR_L for the wide band noise data of Miller is shown in the lower-left panel.

For noise below 55 dB SL the loudness signal-to-noise ratio $\mathrm{SNR}_L \equiv L/\sigma_L$ increases as the cube root $(1-2/3 = 1/3)$ of the loudness; namely the noise increases by a factor of 2 when the loudness increases by a factor of 8. For levels above about 55 dB SL, $\mathrm{SNR}_L(L)$ remains approximately constant with a value between 20 and 60 for both tones and noise. For tones, between threshold and 60 dB SL $\sigma_L \propto L^{1/p}$ with $2 \leqslant p \leqslant 3$. Above 60 dB SL, $\sigma_L \propto L$ (i.e., $p=1$).

To the extent that the curves are all approximately the same across frequency, Fig. 4 provides a stimulus independent description of the relation between the intensity JND and loudness. This invariance in SNR_L seems significant. Where the high level segment of SNR_L is constant, the intensity resolution of the auditory system has a fixed internal *relative* resolution (Ekman, 1959). The obvious interpretation is that as the intensity is increased from threshold, the neural rate-limited SNR increases until it saturates due to some *other* dynamic range limit, such as that due to some form of central nervous system (CNS) noise.

Near-miss to Stevens' law. In Fig. 5 we show a summary of $L(I)$, $\nu(I)$, $J(I)$, and $\Delta L/L = d'/\mathrm{SNR}_L$ for the tone and noise data. For tones the intensity exponent $\nu(I)$ varies systematically between 0.3 and 0.4 above 50 dB SL, as shown by the solid line in the upper-right panel. We have highlighted this change in the power law with intensity for a 1 kHz tone in the upper-right panel with a light-solid straight line. It is logical to call this effect the *near-miss to Stevens' law*, since it cancels the near-miss to Weber's law, giving a constant relative loudness JND $\Delta L/L$ for tones.

In the lower-right panel we provide a functional summary of $\Delta L/L$ for both tones and noise with the light-solid line described by

$$\frac{\Delta L(L)}{L} = h[\min(L,L_0)]^{-1/2}, \tag{37}$$

where $h=\sqrt{2}$ and $L_0 = 5000$ LU (≈ 5 sones). We call this relation the Saturated Poisson Internal Noise (SPIN) model. With these parameter values, Eq. (37) appears to be a lower bound on the relative loudness JND_L for both tones and noise.

E. Weber-fraction formula

In this section we derive the relation between the Weber fraction $J(I)$ given the loudness $L(I)$ starting from the *small-JND approximation*

$$\Delta L = \Delta I L'(I), \tag{38}$$

where $L'(I) \equiv dL/dI$. If we solve this equation for ΔI and divide by I we find

$$J(I) \equiv \frac{\Delta I}{I} = \frac{\Delta L}{IL'(I)}. \tag{39}$$

Finally we substitute the SPIN model Eq. (37)

$$J(I) = \frac{hL(I)}{IL'(I)}[\min(L(I),L_0)]^{-1/2}. \tag{40}$$

This formula is the same as that derived by Hellman and Hellman (1990) when $L \leqslant L_0$. In Fig. 6 we plot Eq. (40) in the lower two panels labeled ''SPIN model.'' From the lower-left panel of this figure, $h=2.4$ and $L_0 = 10\,000$ LU. For levels between 0 and 100 dB SL, the SPIN model (solid curve) fit to Riesz's data and Riesz's formula is excellent. Over this 100 dB range the curve defined by the loudness function fits as well as the curve defined by Riesz's formula given in Appendix A (the dashed curve). The excellent fit gives us further confidence in the basic assumptions of the model.

In the lower-right panel we have superimposed the JND data of Jesteadt *et al.* (1977) with $h=3$ and $L_0 = 10\,000$ LU for comparison to Eq. (40). The Jesteadt *et al.* data were taken with gated stimuli (100% modulation) and 2AFC methods. It is expected that the experimental method would lead to a different value of h than the valued required for Riesz's data set. The discrepancy between 0 and 20 dB may be due to the 100% modulation for these stimuli. The fit from 20 to 80 dB SL is less than a 5% maximum error, and much less in terms of rms error. Note the similarity in slope between the model and the data.

F. Riesz's counting ratio

According to Eq. (31), the frequency dependence of the number of intensity JNDs between any two values of loudness must be isolated to the coefficient $h(f)$. This was first observed empirically by Riesz in 1933 in a different form when he pointed out that for levels below approximately 70 dB SL the JND counting-ratio

$$N_{1X}/N_{1R} \tag{41}$$

is independent of frequency (Riesz, 1933; Houtsma *et al.*, 1980). In this equation, N_{1X} and N_{1R} are given by Eq. (23). The index 1 corresponds to the threshold intensity $I_1 = I_p^*(0)$, the R index indicates some reference intensity I_R [e.g., $L_R(I_R) = 1$ sone at $I_R = 40$ phons], while X indicates an arbitrary intensity I_X. The data of Fig. 2 show a slight frequency dependence of $h(f)$ on f. In the ratio given by Eq. (41), this dependence cancels, making the counting ratio independent of frequency.

Riesz's observation about the JND counting ratio is interesting because the isoloudness contours depend significantly on frequency, $\Delta I(I,f)$ depends significantly on frequency, and yet the ratio Eq. (41), which depends only on $\Delta I(I,f)$, shows little variation with frequency.

By assuming that the counting ratio is independent of frequency, Riesz was able to mimic Munson's loudness curves (Munson, 1932) (i.e., the Fletcher–Munson isoloudness curves) below a critical level of approximately 70 dB SL, given two isolevel contours (e.g., L_1 and L_R) and his 1928 measurements of $\Delta I(I)$ expressed in terms of N using Eq. (23).

From ratio Eq. (41) and Eq. (31) we obtain

$$\frac{N_{1X}}{N_{1R}} = \frac{\sqrt{L_X} - \sqrt{L_1}}{\sqrt{L_R} - \sqrt{L_1}}. \tag{42}$$

Thus we see that the frequency independence of Eq. (42) [namely Riesz's (1933) observation] follows directly from Eq. (32) and the definition of the number of JNDs Eq. (24).

A more general statement may be made. If $\Delta L(L,f)$ has no direct dependence on intensity and is either independent of frequency or contains a frequency dependence which is

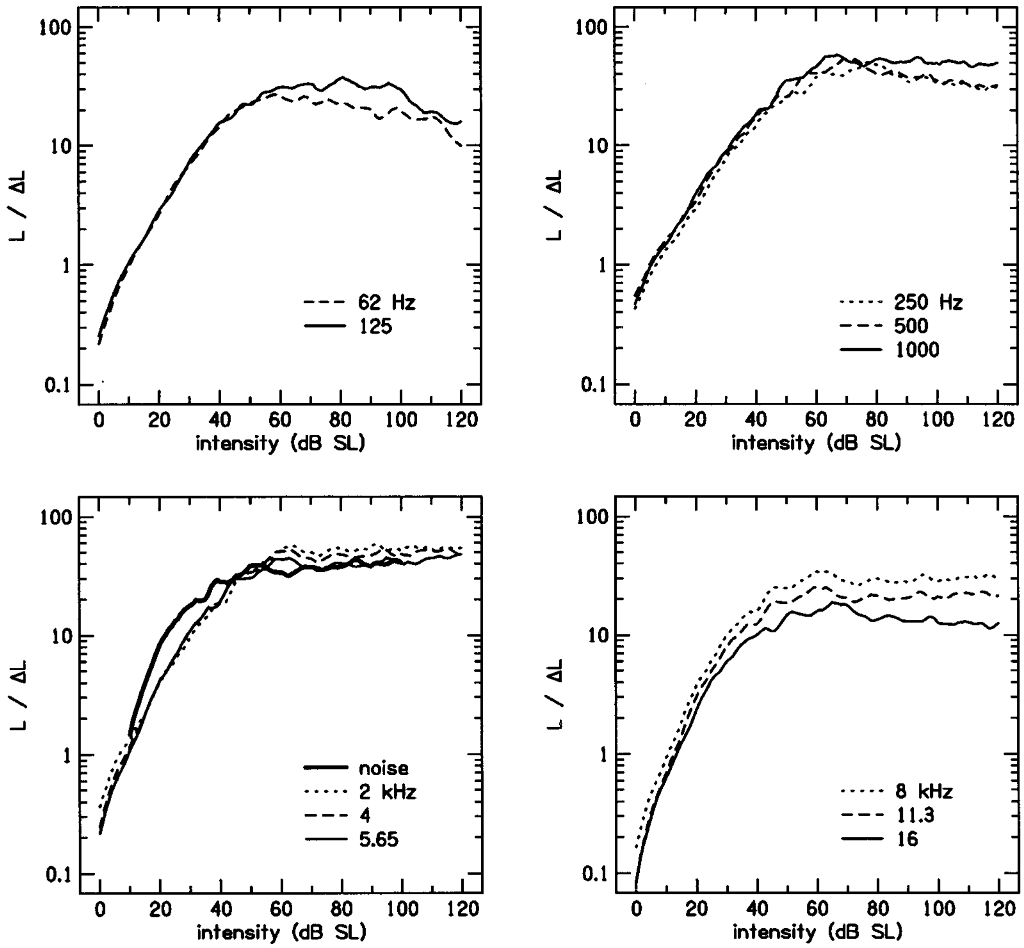

FIG. 4. In this figure we plot $L(I)/\Delta L = \mathrm{SNR}_L/d'$ for intensities between 0 and 120 dB SL. Below about 55 dB SL the internal signal-to-noise ratio $\mathrm{SNR}_L(I)$ is increasing and is proportional to $L^{1-1/p}$, where $2 \leqslant p \leqslant 3$ for tones and $p \approx 3/2$ for noise. Above 60 dB SL the SNR_L saturates at about 50 linear units. At 62 and 125 Hz the SNR_L decreases at high levels.

separable [i.e., $\Delta L(L,f) = \varphi_1(L,\not{I})\varphi_2(f,\not{I})$], then the frequency independence of Riesz's counting ratio follows from Eq. (24), regardless of the detailed form of the dependence of ΔL on L.

G. Summary

Riesz's (1933) observation that the counting ratio is independent of frequency for intensities below 70 dB SL tells us that the loudness JND has no direct dependence on intensity [i.e., $\Delta L(L,\not{I},f)$], and that its dependence on loudness can be separated from any possible dependence on frequency. Turning the argument around, when Riesz's counting ratio is independent of intensity, it follows that $\Delta L(L,\not{I},f)$ (i.e., that ΔL does not depend on I). This observation supports Fechner's idea that $L(I)$ may be found by counting JNDs; he simply had the wrong formulas for $\Delta I(I)$ and $\Delta L(L)$.

IV. A MODEL OF TONAL LOUDNESS CODING

In this section the SPIN model [Eq. (37)] is merged with Fletcher's loudness theory. Fletcher was the first to describe the neural excitation pattern model of partial loudness and propose that the summation of the total spike activity could account for the loudness. Since the variance of the spike count is equal to the mean count for a Poisson process (the PIN model), Fletcher's neural rate model of loudness predicts the JND when the neural spike train obeys Poisson statistics. Above 60 dB, where the SNR_L is saturated, a different explanation is required (e.g., CNS noise).

A. Assumptions about loudness for pure tones

To understand all these relations we need a model, and we make the following model assumptions about the single-trial pure-tone loudness:

(1) The *single-trial pure-tone loudness* $\widetilde{L}(I,f)$ is given by the total number of neural spikes that result from the presentation of the tone of duration $\mathscr{T}$ seconds. Namely

$$\widetilde{L}(I,t,f) = \int_0^{x_L} \int_{t-\mathscr{T}}^{t} \widetilde{\mathscr{R}}(I,t,f,x)\,dt\,dx,$$

where $\widetilde{\mathscr{R}}(I,t,f,x)$ is a random variable that describes the neural spike rate at time t associated with place x on the basilar membrane, given a tone of frequency f and intensity I. The length of the basilar membrane is x_L. The additivity (i.e., the integral over place and time) is based on Fletcher's analysis of 10 years of loudness measurements by Munson

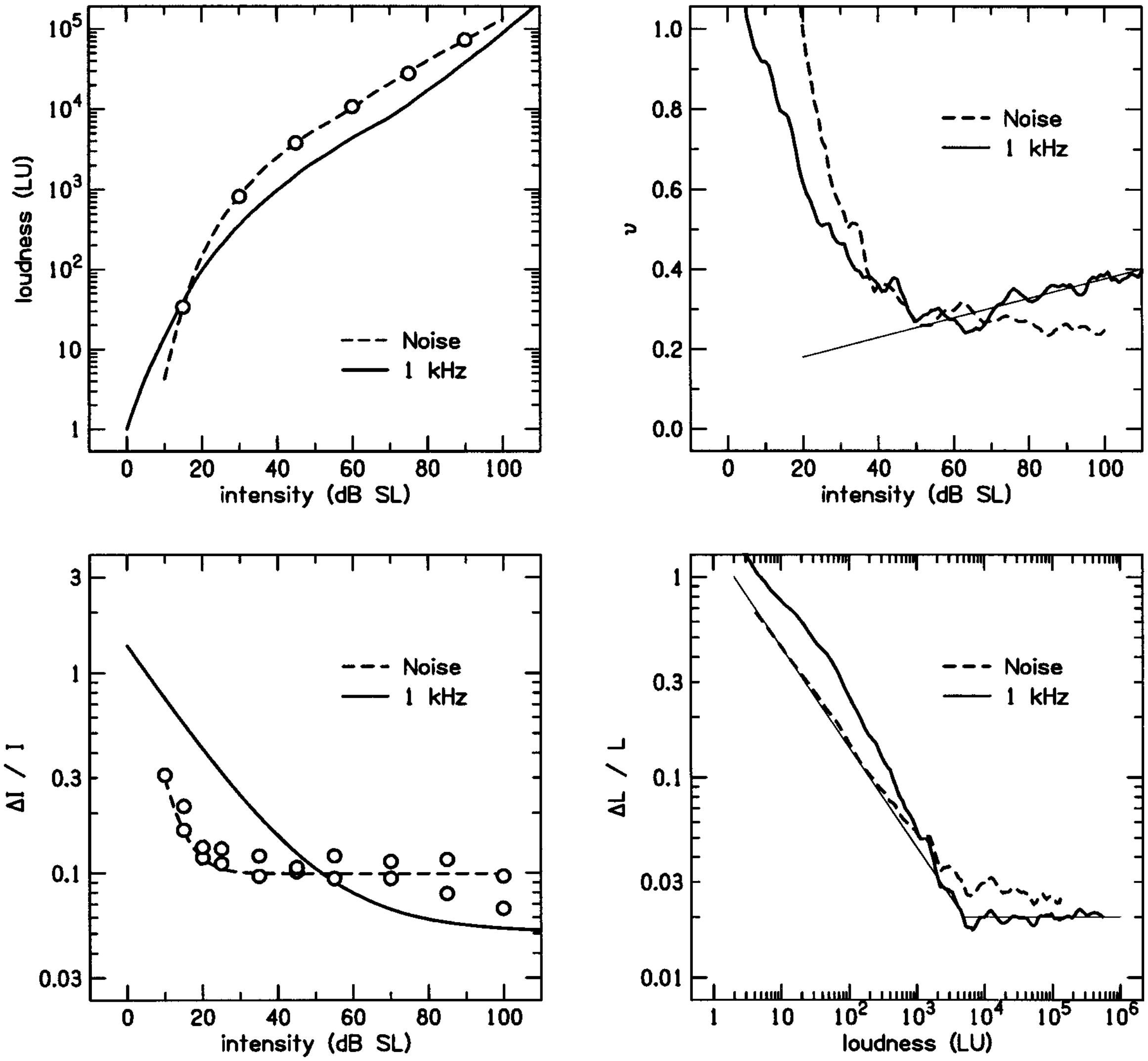

FIG. 5. In 1947 Miller measured the JND_I and the loudness level for two subjects using wide band noise (0.15–7 kHz) for levels between 3 and 100 dB SL. The intensity of the noise was modulated with a ramped square wave that was high for 1.5 s and low 4.5 s. The loudness, computed from Miller's phon data (dashed curve) using Fletcher and Munson's (1933) 1-kHz tone loudness-growth curve are shown in the upper-left panel, along with the Fletcher Munson tonal loudness-growth function (solid curve). The upper-right panel shows the exponent $\nu(I) \equiv dL_{\log}/d\beta$ for both Fletcher and Munson's and Miller's (average of two subjects) loudness-growth function. In the lower-left panel we plot $\Delta I/I$ vs I for Miller's two subjects, Miller's equation, and Riesz's equation. In the bottom-right panel we show the $\Delta L/L$ vs L for the noise and tones cases. From Eq. (20) $\Delta L/L = \nu(I)J(I)$. Note how the product of $\nu(I)$ and $J(I)$ is close to a constant for tones above 65 dB SL. This invariance justifies calling the variations in the power-law exponent $\nu(I)$ for tones the ''near-miss to Stevens' law.'' For reference, 1 sone is 975 LU.

(Fletcher and Munson, 1933), as well as more recent observations.

(2) From signal detection theory, the relation between the loudness JND ΔL and the standard deviation of the single-trial loudness [Eq. (10)] is $\Delta L = d'\sigma_L$.

(3) The single-trial loudness is Poisson below 60 dB SL. A second independent noise source limits the $L/\Delta L$ ratio to a fixed maximum of about 50 for levels above 60 dB.

(4) The loudness-growth function $L(I,f)$ has a slope dL/dI which is a good local approximation to the ratio of the loudness JND to the intensity JND $\Delta L/\Delta I(I,f)$.

B. Model discussion

Fletcher's model (assumption 1) has been heavily and widely criticized (e.g., Licklider, 1959). Clearly, the auditory nerve response is the input to such loudness calculations, but the auditory nerve response may not be claimed to be loudness *per se*. We have shown in this paper that a point-process representation of loudness appears to be a realistic assumption. It is remarkable, given the primitive state of knowledge in 1923 about auditory neurophysiology, that Fletcher associated neural rate with loudness (Fletcher, 1923a; Fletcher and Steinberg, 1924). Unfortunately this association receives only tenuous acceptance today (Viemeister, 1988; Smith, 1988; Delgutte, 1995; Doucet, 1995). Our assumption of a uniform time weighting having duration T is not realistic, and a more realistic weighting function needs further study.

Assumption 2 is widely accepted, and works well for the 2AFC JND task, but is not correct for the modulation-detection task such as Riesz's method of beats. When modulation detection is the task, $\Delta L \approx 0$. This is best seen from Eq. (8). Since α_* is small, the mean change in intensity

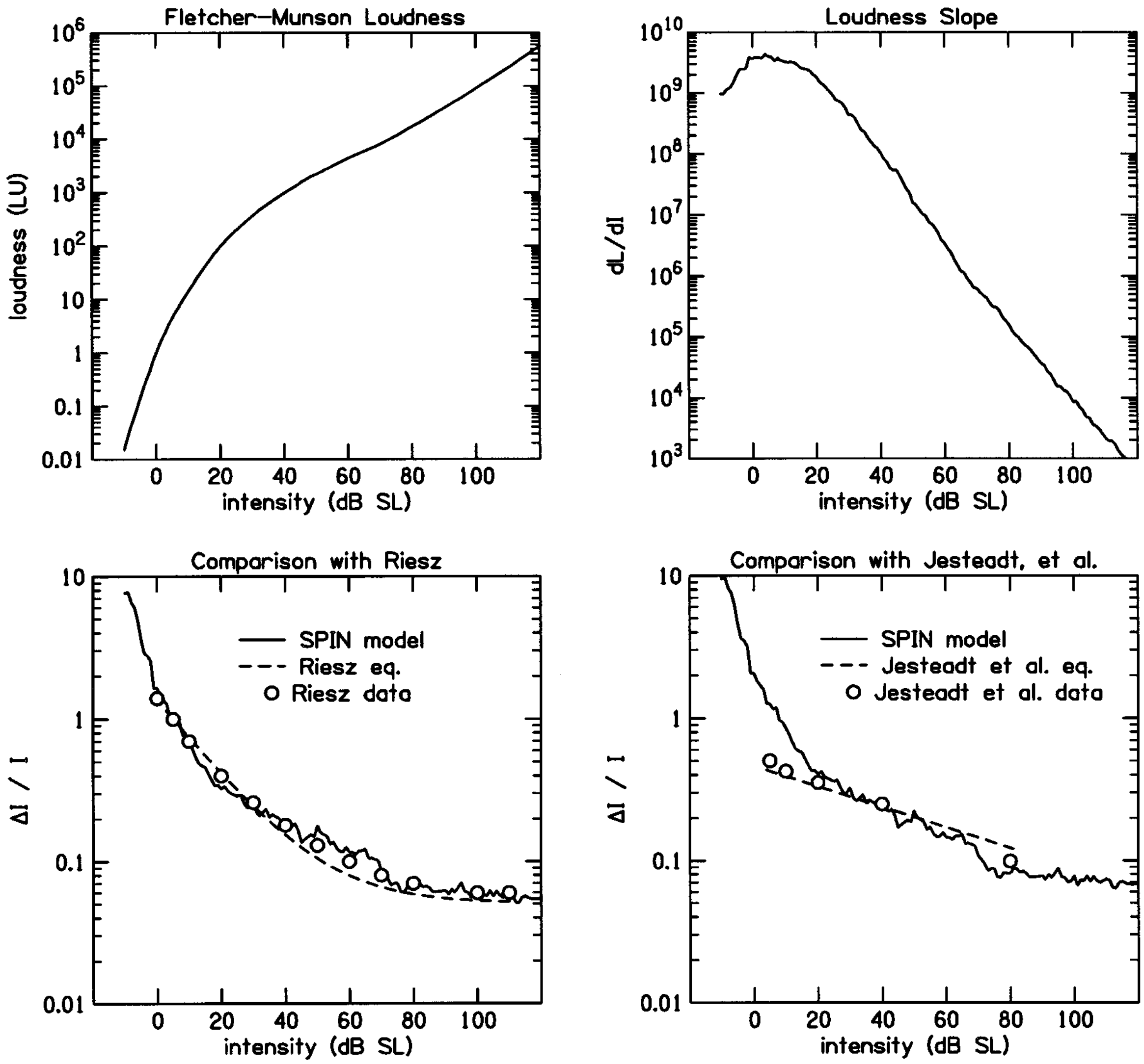

FIG. 6. Comparison between loudness data and intensity JND data at 1 kHz using the SPIN model. The upper-left panel shows the Fletcher–Munson loudness data from their Table III (Fletcher and Munson, 1933). The upper-right panel is a plot of the slope of the loudness with respect to intensity (LU-cm/W). In the lower-left we show the relation between the SPIN-model [Eq. (40) with $h=2.4$] relative JND (solid line), calculated from the Fletcher–Munson loudness data, and the measured relative JND obtained by Riesz (1928) at 1 kHz. We display both Riesz's formula (dashed line) and Riesz's raw data (circles), which may be found in Fletcher (1953, 1995). In the lower right we compare the SPIN-model relative JND [Eq. (40), with $h=3.0$], and the relative JND computed from the Jesteadt *et al.* formula (dashed line) and data from their Table B-I (circles). They measured the JND using pulsed tones for levels between 5 and 80 dB. For reference, 1 sone is 975 LU.

defined by the second term $\alpha_*^2 I$ is not what is detected by the listener. The beating term $2\alpha_* \sin(2\pi f_b t)$ is responsible for detection. From basic detection theory we know that the width of the distribution is responsible for modulation detection rather than the change in the mean. Riesz avoided this problem with the empirical definition of J described in Appendix A.

Assumption 4 is easily tested by direct comparison of the two sides of Eq. (15).

V. DISCUSSION

Inspired by the Poisson internal noise (PIN) based theory of Hellman and Hellman (1990), we have developed a theoretical framework that can be used to explore the relationship between the pure-tone loudness and the intensity JND. The basic idea is to combine Fletcher's neural excitation response pattern model of loudness with signal detection theory. We defined a random decision variable called the single-trial loudness. The *mean* of this random variable is the loudness, while its *standard deviation* is proportional to the loudness JND. We define the loudness signal-to-noise ratio SNR_L as the ratio of loudness (the signal) to standard deviation (a measure of the noise).

A. Model validation

To evaluate the model we have compared the loudness data of Fletcher and Munson (1933) with the intensity JND data of Riesz (1928) for tones. A similar comparison was made for noise using loudness and intensity JND data from Miller (1947). We were able to unify the tone and noise data by two equivalent methods. First, since the loudness SNR is proportional to the ratio of the loudness to the JND $L/\Delta L$, the SNR is also a piecewise power-law function we call the SPIN model. All the data are in excellent agreement with the

SPIN model, providing support for the validity of this theory. Second, we found that the loudness JND $\Delta L(L)$ is a piecewise power law, namely

$$\Delta L(L) = L^{1/p},$$

where p is a piecewise intensity-independent constant. Next we discuss the various piecewise regions for long-duration tones.

1. Below 20 dB SL

In this intensity range we have found that the pure-tone loudness JND is proportional to the square root of the loudness, that is $p=2$. One interpretation of this dependence is that the single-trial loudness obeys Poisson statistics (the PIN model is valid, which says that $\sigma_L^2 = L$), and that the tonal loudness is the average count of the total number of spikes.

From the data of Figs. 1 and 2, we conclude that the PIN JND counting formula Eq. (31) is in excellent agreement with Riesz's (1928) JND data and Fletcher and Munson's (1933) loudness data between 250 Hz and 16 kHz. We take these results as a direct demonstration of the validity of the theory presented in Sec. IV, which implies that the theory's underlying assumptions are correct. Most important is assumption 1 which says that the loudness is equal to the total neural spike count. This same assumption inspired Fletcher's model of loudness and led to the *loudness unit* (LU) scale, which predates the sone scale by 10 years. However, other than for setting the reference intensity corresponding to unit loudness, Fletcher did not actually use the neural counting assumption in his derivation. The success of the PIN theory supports the view that it is Poisson noise that limits our ability to discriminate pure tone intensity below 70 dB (10 sones). In other words, the source of uncertainty that gives rise to the intensity JND is due to the granularity of the neural spikes in the counting representation of loudness, as reflected by assumption 3.

2. Between 20 and 60 dB SL

In this region, for the tone case, p increases from 2 to 3. We have no way of judging the statistical significance of this change to evaluate the significance of this change in exponent. Is it a result of a spread of the excitation pattern, primary neural saturation, or a more central effect? Could it be an anomaly of Riesz's formula for ΔI, or Fletcher and Munson's 1 kHz loudness-growth curve? The only safe conclusion is that we need more data.

3. Above 60 dB SL

Above 60 dB SL the PIN counting formula Eq. (32) begins to fail—above 80 dB it fails dramatically as p approaches 1. At high rates the variance could depend on ''dead-time'' effects (Teich and Khanna, 1985; Young and Barta, 1986) which introduce a correlation between spikes. One problem with the dead-time model is that it does not seem consistent with a p of 1. A more likely possibility is that this high level ''CNS noise'' is due to the variability in spike amplitude, assuming that the output cell soma voltage is sensitive to the area under each spike input.

The direct estimate of $\Delta L(L)$ from Fig. 4 shows that $\sigma_L(I) \propto L$ (Ekman, 1959), leading to a loudness SNR_L of ≈ 50. We may understand better what is happening in this region by looking at the model. If we combine Eqs. (13) and (18), we find

$$J(I) = \frac{d'}{\nu(I)\ \mathrm{SNR}_L(I)}. \tag{43}$$

From this equation it appears that the near-miss to Weber's law above 60 dB SL results from the variations in $\nu(I)$ with I, since SNR_L is independent of I in this region. We call this small variation in $\nu(I)$ the near-miss to Stevens law.

An example. As a sanity check on Eq. (43), we calculate SNR_L for Miller's wideband JND data. As shown in Fig. 5 lower-left panel, dashed line, Miller[11] found $J=0.1$. From Eq. (13), assuming $d' \approx 1$, SNR_I is therefore 10. As shown in Fig. 5, upper right, the power-law exponent is $\nu \approx 1/4$ at 60 dB SL for noise, which means $\mathrm{SNR}_L \approx 40$. This estimate is in reasonable agreement with the measured values of Fig. 4.

B. The noise model

1. The SPIN model

Equation (37) summarizes our results on the relative loudness JND for both tones and noise. Using this formula along with Eq. (18), the JND may be estimated for tones and noise once the loudness has been determined, by measurement, or by model. Fechner's postulate, that the loudness JND is constant, is not supported by our analysis, in agreement with Stevens (1961).

2. The PIN model

The success of the PIN model is consistent with the idea that the pure-tone loudness code is based on neural discharge rate. This theory should apply between threshold and moderate intensities (e.g., <60 dB) for ''frozen stimuli'' where the JND is limited by internal noise.

3. CNS noise

Above 60 dB SL we find that the loudness signal-to-noise ratio saturated with a constant loudness SNR between 30 and 50 for both the tone and noise conditions, as summarized by Ekman's law (Ekman 1959). We conclude that the Hellman and Hellman theory must be modified to work at these higher intensities.

4. Weber's law

It is significant that while both $J(I)$ and $\nu(I)$ vary with intensity, the product is constant above 60 dB SL. Given that $J = d'/\nu\, \mathrm{SNR}_L$, the saturation in SNR_L explains Weber's law for wideband signals (since ν and SNR_L for that case are constant) as well as the near–miss to Weber's law for tones, where ν is not constant (the near-miss to Stevens' law, Fig. 5).

5. Generalization to other data

If $\sigma_L(L,I)$ depends on L, and is independent of I, then the $\mathrm{SNR}_L(L)$ should not depend on the nature of the function $L(I)$ [i.e., it should be true for any $L(I)$]. This prediction is supported by our analysis summarized by Eq. (37). It will be interesting to see how SNR_L depends on L and I for subjects having a hearing-loss-induced recruitment, and how well this theory explains other data in the literature, such as loudness and JNDs with masking-induced recruitment (Schlauch *et al.*, 1995).

6. Conditions for model validity

To further test the SPIN model, several conditions must be met. First the loudness and the JND must have been measured under the same stimulus conditions. Second, the internal noise must be the dominate factor in determining the JND. This means that the stimuli must be frozen (or have significant duration and bandwidth), and the subjects well trained in the task. As the signal uncertainty begins to dominate the internal noise, as it does in the cases of roving the stimulus, the intensity JND will become independent of the loudness.

As discussed by Stevens and Davis (Stevens and Davis, 1983, pp. 141–143), JND data are quite sensitive to the modulation conditions. The Riesz (1928) and Munson (1932) data make an interesting comparison because they are taken under steady-state conditions and are long duration tonal signals. Both sets of experimental data (i.e., Riesz and Munson) were taken in the same laboratory within a few years of each other.[12] Riesz (1928) states that he used the same methods as Wegel and Lane (1924), and it is likely that Munson (1932) did as well.

Differences in the signal conditions are the most likely explanation for the differences observed in the intensity JND measurements of Riesz and Jesteadt shown in Fig. 6. One difference between the data of Riesz (1928) and Jesteadt *et al.* (1977) is that Riesz varied the amplitude of the tones in a sinusoidal manner with a small (i.e., just detectable) modulation index, while Jesteadt *et al.* alternated between two intervals of different amplitude, requiring that the tones be gated on and off (i.e., a 100% modulation index).

The neural response to transient portions of a stimulus is typically larger than the steady-state response (e.g., neural overshoot) and, therefore, may dominate the perception of stimuli with large abrupt changes in amplitude. The fact that the intensity JND is sensitive to the time interval between two tones of different amplitude (Stevens and Davis, 1983) is another indication that neural overshoot may play a role.

It would be interesting to check the SPIN model on loudness and JND data taken using gated signals, given the observed sensitivity to the modulation. While these JND data are available (Jesteadt *et al.*, 1977), one would need loudness data taken with identical (or at least similar) modulations. We are not aware of such data.

C. Discussion of the model

1. Does Weber's law hold in a single channel?

It has been observed that Weber's law holds for wide-band stimuli (Florentine and Buus, 1981; Viemeister, 1988). This observation has led to the conclusion that Weber's law must hold in a single auditory channel. Because SNR_L is approximately the same for both tones and noise, we are led to the conclusion that the source of noise for Miller's JND experiment is the same internal noise as that for tones. Important questions are: If the noise is internal, and both $\mathrm{SNR}_L(L)$ and $\nu(L)$ depend on L, why is $\nu(L)\,\mathrm{SNR}_L$ constant when many channels are excited? Does this observation hold true for both frozen as well as random stimuli? What is the physical mechanism that determines the value of ν in the normal cochlea?

Miller's data shows that J is constant from 20 to 80 dB SL. Above 80 the relative JND seems to decrease slightly, and below 20 it dramatically increases. Between 20 and 50 dB SL both $\nu(I)$ and $\mathrm{SNR}_L(I)$ change by a factor of 4, but in such a way that their product is constant. While the source of this covariation is presently unknown, it may be related to the compressive role of outer hair cell feedback (Allen, 1996b).

2. Near-miss and the spread of excitation

Based on the results presented here it seems that $\Delta L/L$ is the invariant (Ekman's law) above about 5 sones (5 000 LU) rather than $\Delta I/I$. As a result of Eq. (18), when $\nu(I)$ is constant, Weber's law must hold. In this view, the ''near-miss'' to Weber 1988's law results from the range of $\nu(I,x)$ values that contribute to the specific loudness (i.e., $\mathscr{R}$) for pure tones. If ν were independent of intensity [i.e., if Stevens' law strictly held and $L(I)$ was exactly a power law], the addition of components of differing intensities leads to a power law, that is

$$(I+aI)^\nu=(1+a)^\nu I^\nu. \tag{44}$$

When $\nu(I)$ depends on intensity, the sum is no longer strictly a power law (i.e., the near-miss). According to this view, the near-miss results from the large spread of intensities, and therefore of exponents $\nu(I)$, in the tonal excitation pattern. This explaination seems different than the 1981 spread of masking explanation of the near-miss offered by Florentine and Buus.

3. A correlation with other cochlear measures

It seems to be more than coincidence that 60 dB is where the cochlear microphonic saturates, two-tone suppression neural threshold sets in (Fahey and Allen, 1985), the upward spread of masking becomes important (Wegel and Lane, 1924), and the internal noise of the SPIN model saturates. If the saturation of the SNR_L (Ekman's law) seen in Fig. 4 is found for other experimental conditions, then it is an important result that could lead to a great simplification of our understanding of neural coding. It is important to establish the source of the saturation, which might be viewed as some form of CNS noise. This saturated region, which is an example of Ekman's law, supersedes Weber's law. Ekman's law is similar to Weber's law, but instead of the Φ relative JND being constant, it is the Ψ relative JND that is constant. Schlauch *et al.* (1995) tested Ekman's law and found it did not provide a good fit to their data.

Some measurements of the relative intensity JND_I have shown a discontinuity around 60 dB SL (Rabinowitz *et al.*, 1976; Greenwood, 1993), which is not apparent in the data of Riesz (1928) and Jesteadt *et al.* (1977). This intensity JND_I discontinuity may be related to saturation of the loudness JND_L.

VI. SUMMARY

A summary list of some of the main conclusions of this paper is:

—Fechner's postulate is not valid, except perhaps below 125 Hz and 50 dB SPL.

—Fechner's idea that the *number* of JNDs may be useful in defining a basic psychophysical scale which quantifies supra-threshold loudness seems correct if modified to allow ΔL to depend on L.

—Once $\Delta I(I)$ and $\Delta L(L)$ are known, $L(I)$ may be determined from Eq. (22).

—The near-miss to Weber's law for tones covaries with the near-miss to Stevens' law, defined as a deviation from a power-law dependence of loudness on intensity for tones.

—Above 125 Hz, a possible replacement for Fechner's law for tones is given by Eq. (36). This formula assumes that $\Delta L = \sqrt{L}$, and therefore should be valid for tones between 0 and 60 dB SL.

—The variance of the single-trial loudness is strictly proportional to the mean of the single trial loudness ($\sigma_L^2 = L$) for $I < 20$ dB for frequencies between 250 Hz and 16 kHz.

—The variance of the single-trial loudness is approximately proportional to the mean of the single-trial loudness ($\sigma_L^2 \approx L$) for $I < 60$ dB, for frequencies between 250 Hz and 16 kHz.

—The observation that $\sigma_L^2 \approx L$ for $I < 60$ dB (Fig. 3) is not inconsistent with the near-miss to Weber's law for tones or Weber's law for wide band stimuli.

—The standard deviation of the single trial loudness is proportional to the mean of the single-trial loudness ($\sigma_L \propto L$) for $I > 60$ dB SL, for all frequencies.

—The PIN model is easily merged with Fletcher's neural counting model of loudness.

—At 1 kHz the loudness SNR_L of the auditory system seems to saturate at a value of ≈ 50 (linear units) at an intensity of ≈ 60 dB SL.

—When $L(I) \propto I^\nu$, $\mathrm{SNR}_I = \nu \mathrm{SNR}_L$.

—When $L(I) \propto I^\nu$, the Weber fraction is $J = d'/(\nu\, \mathrm{SNR}_L)$.

—We interpret the invariance of Riesz's JND counting ratio with frequency in terms of Eq. (22) as showing that the loudness JND for tones is a separable function of loudness and frequency, and is not a function of intensity [i.e., $\Delta L(f, L, \cancel{I}) = \phi(f)\psi(L)$].

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APPENDIX A: RIESZ'S EXPERIMENT

The Riesz intensity JND data were measured by modulation detection. Two tones, separated by small ''beat'' frequency difference (e.g., 3 Hz), were presented to the subject, who was asked to vary the level of the lower-level, higher-frequency tone, until the 3-Hz beat was just detectable. The Weber fraction was computed from the relative levels of the tones using the relation

$$J(I) = \frac{(a_1 + a_2)^2 - (a_1 - a_2)^2}{(a_1 - a_2)^2},$$

where $a_1(f_1)$ and $a_2(f_2)$ are the *peak* amplitudes of the two tones at f_1 and f_2, with $f_2 - f_1 > 0$.

The first series was taken at 25 and 50 dB SL at $f_1 = 1$ kHz for eight beat frequencies ranging from 0.2 to 30 Hz. The best-beats detection frequency was found to be 3 Hz. A second series of measurements was made using a beat frequency of 3 Hz, as a function of frequency f_1 between 35 and 10 000 Hz and levels of a_1 between threshold and 110 dB SL (the upper limit depended on frequency). Twelve male subjects were used.

Riesz summarized his data using a formula for $J(I,f) \equiv \Delta I(I)/I$, that fit the mean data points. This important formula is repeated here for convenience

$$J(I,f) = J_\infty(f) + (J_0(f) - J_\infty(f))(I_0/I)^{\kappa(f)}. \tag{A1}$$

It has three frequency-dependent parameters

$$J_\infty(f) = 15 \times 10^{-6} f + \frac{126}{(80 f^{1/2} + f)}, \tag{A2}$$

$$J_0(f) = 0.3 + 0.3 \times 10^{-3} f + \frac{193}{f^{0.8}}, \tag{A3}$$

$$\kappa(f) = \frac{0.244 \times 10^6}{(0.358 \times 10^6 f^{1/8} + f^2)} + \frac{0.65 f}{(3500 + f)}, \tag{A4}$$

where $J_0(f) = J(I_0, f)$ and $J_\infty(f) = J(I \to \infty, f)$.

Riesz also evaluated the integral for the number of JNDs between I_0 and I_1 Eq. (23),

$$N_{01}=\frac{1}{\kappa(f)J_\infty(f)}\ln\left(\frac{J_\infty(f)(I_1/I_0)^{\kappa(f)}+(J_0(f)-J_\infty(f))}{J_0(f)}\right). \tag{A5}$$

It is interesting to compare Riesz's N_{01} to Eq. (26). A table of $\Delta a=a_1-a_2$ values (i.e., Riesz's raw data) may be found in Fletcher (1953, Table 24, page 146).

APPENDIX B: RECONSTRUCTION OF THE LOUDNESS CURVES

The isoloudness data were first reported in 1932 by Munson at 11 different frequencies, for 11 subjects, using earphones (Munson, 1932; Fletcher and Munson, 1933). In this Appendix we describe the procedure and the assumptions required to reconstruct the loudness-growth curves $L(I,f)$ at any frequency f (Jeng, 1992).

It is helpful to have a notation to describe the isoloudness curves. The pressure of a test tone at test frequency f is defined as $P(f)$. The reference frequency is 1 kHz and the reference tone pressure is $P_r=P(1000)$. The average hearing threshold at the reference frequency is defined as P_0. We use the superscript asterisk on $P(f)$ to indicate that the test tone pressure corresponds to the isoloudness condition. Thus $P^*(P_r,f)$ is defined by the condition $L(P^*,f)=L(P_r,1000)$, which says that the *loudness* of the test tone at frequency f and pressure $P^*(P_r,f)$ is equal to the loudness of the reference tone at 1 kHz. Equally loud sounds define the *phon* scale of *loudness level*. Thus $P^*(P_r,f)$ is said to be at 20 $\log_{10}(P_r/P_0)$ phons. Loudness level, in phons, is *not* a loudness scale (loudness is measured is *sones*).

The raw data are given in Table I of Fletcher and Munson (1933), which provides rms voltages on the earphone at ten frequencies from 62 Hz to 16 kHz, expressed in dBV. Since the earphone is linear, the voltage across the earphone and the ear canal pressure are related by a scale factor. Thus the values (Fletcher and Munson, 1933, Table I) provide estimates of 20 $\log_{10}(P^*(P_r,f)/P_r)$, namely the relative intensity of a test tone in dB that is equal in loudness to the reference tone. We have reduced this data to a frequency dependent regression. Thus to find the phon value $P^*(P_r,f)$ at frequency f, one may use the regression coefficients of our Table BI, derived in the next section.

Fletcher and Munson's Table III gives the *loudness* $G(P_r)\equiv L(1000,P_r)$, which is plotted in the upper-left panel of our Fig. 6. Today loudness is defined using the sone scale. One sone is the loudness at 40 dB SPL at 1 kHz. In 1933 Fletcher and Munson used the Loudness-Unit (LU) scale. One LU is the loudness at 0 dB SL at 1 kHz. These scales differ in ratio by 975, namely 975 LUs is 1 sone.

To compute $L(P,f)$ for any f and $P(f)$ there are two ways to proceed. The first method is to compute $G(P_r)$ and $P^*(P_r,f)$ using the reference pressure P_r as the independent variable. One can then plot G vs P^*. While this method is simple, it does not directly give the loudness for any value of $P(f)$, as P_r is the independent variable, but can be used to build a ''look-up'' table. The second method, which is logically more direct, is to define the *inverse* of the phon function $P^*(P_r,f)$ corresponding to the test pressure $P(f)$, which we define as $P_r^*(P,f)$. Using this notation, $G(P,f)=L(P_r^*(P,f))$. If you think this is confusing, you have a greater than average attention span. We used the first method. Linear interpolation was then used to obtain the loudness for phon values between the tabulated values, which were computed in 1-dB steps from -10 to 129 dB.

TABLE BI. Regression coefficients used to calculate the phon value at the frequency defined by the first column. This regression relates the ear canal sound pressure in dB at 1 kHz to the ear canal sound pressure in dB at the measurement frequency, that is equally loud, as measured by Munson (1932) and Fletcher and Munson (1933).

f	c_3	c_2	c_1	c_0
0.062	7.46169e-05	−0.00984189	0.74629	0.425879
0.125	5.2594e-05	−0.00654132	0.800557	0.295663
0.25	0	0.00124457	0.720323	0.780066
0.5	0	0.00209933	0.761911	0.467849
1	0	0	1	0
2	0	−0.0011956	1.14141	−0.622967
4	0	−0.00240718	1.2314	−0.393083
5.65	0	−0.00272458	1.24014	0.481007
8	0	−0.00232339	1.20659	0.0426691
11.3	0	−0.002439	1.24474	−1.51871
16	0	−0.000566296	1.03446	−1.82771

1. Phon estimation

We used polynomial regression (Jeng, 1992, p. 27) on the raw data (Fletcher and Munson, 1933, Table I) of isoloudness measurements to define $P^*(f,P_r)$. The measurements of the subject's threshold, given in the lower portion of their Table I, were also used in the regression to increase the accuracy of regression estimate at threshold. It was also important to use the 1-kHz reference values as the abscissa, when setting up the regression, since these are unaffected by the subject's loudness estimate variability (Jeng, personal communication).

The resulting regression coefficients, given in Table BI, are defined by the cubic polynomial

$$y=c_3x^3+c_2x^2+c_1x^1+c_0,$$

where the abscissa $x=20\ \log_{10}(P_r/P_0)$ represents the 1-kHz reference earphone voltage in dB and the ordinate $y=20\ \log_{10}(P^*/P_0)$ is the earphone voltage at the frequency where the phon value is being specified. For the two lowest frequencies, at 62 and 125 Hz, it was necessary to use third-order polynomials, while second-order regressions were adequate for the remaining frequencies.

[1]It may be helpful to note that Φ and Ψ sound similar to the initial syllable of the words *phy*sical and *psy*chological, respectively (Boring, 1929).

[2]The symbol $\equiv$ denotes ''equivalence.'' It means that the quantity to the left of the $\equiv$ is defined by the quantity on the right.

[3]Equivalence of the pressure and intensity references requires that $\varrho c=40$ cgs Rayls. At standard atmospheric pressure, this is only true when the temperature is about 39 °C.

[4]Because of small fluctuations in I_m and I_p due to the finite integration time T, this equality cannot be exactly true. We specifically ignore these small rapid fluctuations—when these rapid fluctuations are important, our conclusions and model results must be reevaluated.

[5]It is traditional to define the intensity JND to be a function of I, rather than a function of $\alpha(I)$, as we have done here. We shall treat both notations as equivalent [i.e., $\Delta I(I)$ or $\Delta I(\alpha(I))$].

[6]As a mnemonic, think of the $\sim$ as a ''wiggle'' associated with randomness.

[7]We are only considering the auditory case of Fechner's more general theory.

[8]Except, as we shall show, in the limited region below 125 Hz and 50 dB SL.

[9]For example, when the signal is roved, the JND will be determined by the magnitude of the rove, and the loudness and the JND must be independent.

[10]W. Siebert, personal communication.

[11]Miller used 10 $\log(1+J)$ as the measure of the JND rather than J.

[12]In 1928 Wegel, Riesz, and Munson were all members of Fletcher's department.

Allen, J. B. (**1996a**). ''Harvey Fletcher's role in the creation of communication acoustics,'' J. Acoust. Soc. Am. **99**, 1825–1839.

Allen, J. B. (**1996b**). ''OHCs shift the excitation pattern via BM tension,'' in *Diversity in Auditory Mechanics*, edited by E. Lewis, G. Long, R. Lyon, P. Narins, and C. Steel (World Scientific, Singapore).

Allen, J. B. (**1995**). ''Harvey Fletcher 1884–1981,'' in The ASA edition of *Speech, Hearing in Communication*, edited by J. B. Allen (Acoustical Society of America, Woodbury, New York).

Boring, E. (**1929**). *History of Psychophysics* (Appleton–Century, New York).

De Vries, H. (**1943**). ''The quantum character of light and its bearing upon the threshold of vision, the differential sensitivity and the acuity of the eye,'' *Physica* **10**, 553–564.

Delgutte, B. (**1995**). ''Physiological models for basic auditory percepts,'' in *Auditory Computation*, edited by H. Hawkins and T. McMullen (Springer-Verlag, New York).

Doucet, J. R. (**1995**). ''The perstimulus compound action potential and intensity coding in the chinchilla auditory nerve,'' Technical Report No. ISR-S-32, Institute for Sensory Research, Syracuse University, Syracuse, New York.

Egan, J., and Hake, H. (**1950**). ''On the masking pattern of a simple auditory stimulus,'' J. Acoust. Soc. Am. **22**, 662–630.

Ekman, G. (**1959**). ''Weber's law and related functions,'' Psychology **47**, 343–352.

Fahey, P. F., and Allen, J. B. (**1985**). ''Nonlinear phenomena as observed in the ear canal, at the auditory nerve,'' J. Acoust. Soc. Am. **77**, 599–612.

Fechner, G. (**1966**). ''Translation of: Elemente der psychophysik,'' in *Elements of Psychophysics, Volume I*, edited by H. Adler (Holt, Rinehart, and Winston, New York).

Fletcher, H. (**1923a**). ''Physical measurements of audition and their bearing on the theory of hearing,'' J. Franklin Inst. **196**, 289–326.

Fletcher, H. (**1923b**). ''Physical measurements of audition and their bearing on the theory of hearing,'' Bell Syst. Tech. J. **ii**, 145–180.

Fletcher, H. (**1929**). *Speech and Hearing* (Van Nostrand, New York).

Fletcher, H. (**1953**). *Speech and Hearing in Communication* (Krieger, Huntington, NY).

Fletcher, H. (**1995**). ''Speech and Hearing in Communication,'' in *The ASA edition of Speech and Hearing in Communication*, edited by J. B. Allen (Acoustical Society of America, New York).

Fletcher, H., and Munson, W. (**1933**). ''Loudness, its definition, measurement, and calculation,'' J. Acoust. Soc. Am. **5**, 82–108.

Fletcher, H., and Steinberg, J. (**1924**). ''The dependence of the loudness of a complex sound upon the energy in the various frequency regions of the sound,'' Phys. Rev. **24**, 306–317.

Florentine, M., and Buus, S. (**1981**). ''An excitation-pattern model for intensity discrimination,'' J. Acoust. Soc. Am. **70**, 1646–1654.

Gescheider, G. (**1976**). *Psychophysics: Method and Theory* (Erlbaum, Hillsdale, NJ).

Green, D. (**1970**). ''Application of detection theory in psychophysics,'' Proc. IEEE **58**, 713.

Green, D. (**1988**). ''Audition: Psychophysics and perception,'' in *Stevens' Handbook of Experimental Psychology*, edited by R. Atkinson, R. Herrnstein, G. Lindzey, and R. Luce (Wiley, New York), Chap. 6, pp. 327–376.

Green, D., and Swets, J. (**1966**). *Signal Detection Theory and Psychophysics* (Wiley, New York).

Greenwood, D. (**1993**). ''The intensitive DL of tones: Dependence of signal/masker ratio on tone level and on specturm of added noise,'' Hearing Res. **65**, 1–39.

Hellman, W., and Hellman, R. (**1990**). ''Intensity discrimination as the driving force for loudness. Application to pure tones in quiet,'' J. Acoust. Soc. Am. **87**, 1255–1271.

Houtsma, A., Durlach, N., and Braida, L. (**1980**). ''Intensity perception XI. Experimental results on the relation of intensity resolution to loudness matching,'' J. Acoust. Soc. Am. **68**, 807–813.

Jeng, P. (**1992**). ''Loudness predictions using a physiologically based auditory model,'' Ph.D. thesis, City University of New York, Ann Arbor, MI.

Jeng, P. (personal communication).

Jesteadt, W., Wier, C., and Green, D. (**1977**). ''Intensity discrimination as a function of frequency and sensation level,'' J. Acoust. Soc. Am. **61**, 169–177.

Johnson, J., Turner, C., Zwislocki, J., and Margolis, R. (**1993**). ''Just noticeable differences for intensity and their relation to loudness,'' J. Acoust. Soc. Am. **93**, 983–991.

Licklider, J. (**1959**). ''Three auditory theories,'' in *Psychology: A Study of a Science. Study I, Volume 1*, edited by S. Koch (McGraw-Hill, New York), pp. 41–144.

Littler, T. (**1965**). *The Physics of the Ear* (Pergamon, Oxford, England).

Luce, R. (**1993**). *Sound and Hearing* (Earlbaum, Hillsdale, NJ).

Marks, L. (**1974**). *Sensory Processes: The New Psychophysics* (Academic, New York).

McGill, W., and Goldberg, J. (**1968a**). ''Pure-tone intensity discrimination as energy detection,'' J. Acoust. Soc. Am. **44**, 576–581.

McGill, W., and Goldberg, J. (**1968b**). ''A study of the near-miss involving Weber's law and pure tone intensity discrimination,'' Percept. Psychophys. **4**, 105–109.

Miller, G. A. (**1947**). ''Sensitivity to changes in the intensity of white noise and its relation to masking and loudness,'' J. Acoust. Soc. Am. **19**, 609–619.

Montgomery, H. (**1935**). ''Influence of experimental technique on the measurement of differential intensty sensitivity of the ear,'' J. Acoust. Soc. Am. **7**, 39–43.

Moore, B. (**1982**). *An Introduction to the Psychology of Hearing* (Academic, London), 2nd ed.

Munson, W. (**1932**). ''An experimental determination of the equivalent loudness of pure tones,'' J. Acoust. Soc. Am. **4**, 7 (Abstract).

Munson, W. (**1947**). ''The growth of auditory sensation,'' J. Acoust. Soc. Am. **19**, 584–591.

Nutting, P. (**1907**). ''The complete form of Fechner's law,'' Bull. Bureau Stand. **3**, 59–64.

Plack, C., and Carlyon, R. (**1995**). ''Loudness perception and intensity coding,'' in *Hearing, Handbook of Perception and Cognition*, edited by B. Moore (Academic, San Diego), Chap. 4, pp. 123–160.

Raab, D., and Goldberg, I. (**1975**). ''Auditory intensity discrimination with bursts of reproducible noise,'' J. Acoust. Soc. Am. **57**, 473–447.

Rabinowitz, W., Lim, J., Braida, L., and Durlach, N. (**1976**). ''Intensity perception: VI. Summary of recent data on deviations from Weber's law for 1000-Hz tone pulses,'' J. Acoust. Soc. Am. **59**, 449–450.

Riesz, R. (**1928**). ''Differential intensity sensitivity of the ear for pure tones,'' Phys. Rev. **31**, 867–875.

Riesz, R. (**1933**). ''The relationship between loudness and the minimum perceptible increment of intensity,'' J. Acoust. Soc. Am. **4**, 211–216.

Schlauch, R., Harvey, S., and Lanthier, N. (**1995**). ''Intensity resolution and loudness in broadband noise,'' J. Acoust. Soc. Am. **98**, 1895–1902.

Siebert, W. (**1965**). ''Some implications of the stochastic behavior of primary auditory neurons,'' Kybernetik **2**, 205–215.

Siebert, W. (**1968**). ''Stimulus transformations in the peripheral auditory system,'' in *Recognizing patterns*, edited by P. Kolers and M. Eden (MIT, Cambridge, MA), Chap. 4, pp. 104–133.

Siebert, W. (personal communication).

Smith, R. L. (**1988**). ''Encoding of sound intensity by auditory neurons,'' in *Auditory Function*, edited by G. Edelman, W. Gall, and W. Cowan (Wiley, New York), Chap. 7, pp. 243–274.

Steinberg, J. (**1925**). ''The loudness of a sound and its physical stimulus,'' Phys. Rev. **26**, 507.

Stevens, S. (**1951**). ''Mathematics, measurement, and psychophysics,'' in *Handbook of Experimental Psychology*, edited by S. Stevens (Wiley, New York), Chap. 1, pp. 1–49.

Stevens, S. (**1961**). ''To honor Fechner and repeal his law,'' Science **133**, 80–86.

Stevens, S., and Davis, H. (**1983**). *Hearing, Its Psychology and Physiology* (The Acoustical Society of America, Woodbury, New York).

Sydorenko, M., and Allen, J. B. (**1994**). ''The relationship between the difference limen (DL) in intensity and the masked threshold of sub-critical bandwidth signals,'' J. Acoust. Soc. Am. **95**, 2942.

Teich, M., and Khanna, S. (**1985**). ''Pulse-number distribution for the neural spike train in the cat's auditory nerve,'' J. Acoust. Soc. Am. **77**, 1110–1128.

Viemeister, N. F. (**1988**). ''Psychophysical aspects of auditory intensity coding,'' in *Auditory Function*, edited by G. Edelman, W. Gall, and W. Cowan (Wiley, New York), Chap. 7, pp. 213–241.

Weber, E. H. (**1988**). ''Der tastsinn und das gemainfül,'' in *Handwörterbuch der Physiologie*, edited by R. Wagner (Vieweg, Braunschweig), Vol. 3, Chap. 7, pp. 481–588.

Wegel, R., and Lane, C. (**1924**). ''The auditory masking of one pure tone by another and its probable relation to the dynamics of the inner ear,'' Phys. Rev. **23**, 266–285.

Yost, W. (**1994**). *Fundamentals of Hearing, An Introduction* (Academic, San Diego).

Young, E. D., and Barta, P. E. (**1986**). ''Rate responses of auditory nerve fibers to tones in noise near masked threshold,'' J. Acoust. Soc. Am. **79**, 426–442.

Zwislocki, J., and Jordan, H. (**1986**). ''On the relation of intensity JNDs to loudness and neural noise,'' J. Acoust. Soc. Am. **79**, 772–780.